Magnetic Symmetry in Path Integrals for Compact Quantum Electrodynamics

Randy S

Abstract Compact electrodynamics (gauged group $U(1)$) without matter in d-dimensional spacetime has a **magnetic $(d-3)$-form symmetry**, an example of a higher-form symmetry (article 09181). The magnetic symmetry is generated by topological Wilson operators localized on two-dimensional surfaces, and the corresponding charged operators are 't Hooft operators localized on closed $(d-3)$-dimensional manifolds.

Articles 40191 and 93302 construct generalizations of these operators for any compact connected gauged group G, treating spacetime as discrete so the math is elementary and unambiguous. This article specializes those constructions to electrodynamics ($G = U(1)$). The relationship between these magnetically charged 't Hooft operators and the topological 't Hooft operators constructed in article 82508 is explained.

Contents

1 Introduction

Article 09181 introduces the concept of **higher-form symmetry**. Compact electrodynamics (gauged group $U(1)$) without matter in d-dimensional spacetime has a pair of higher-form symmetries: an electric one-form symmetry, and a magnetic $(d-3)$-form symmetry.[1] Here's a summary using the notation from article 22721:

- The electric one-form symmetry is generated by topological 't Hooft operators $T^\bullet$ localized on[2] $(d-2)$-dimensional submanifolds of spacetime, and the corresponding charged operators are Wilson operators W° localized on closed one-dimensional manifolds.
- The magnetic $(d-3)$-form symmetry is generated by topological Wilson operators $W^\bullet$ localized on two-dimensional surfaces in spacetime, and the corresponding charged operators are 't Hooft operators T° localized on closed $(d-3)$-dimensional manifolds.

In quantum field theory, most models of interest have not been constructed directly in smooth spacetime, but many models of interest can be constructed when spacetime is (very finely) discretized. In that context, the operators $T^\bullet$ and W° involved in the electric one-form symmetry are relatively easy to describe.[3,4] In contrast, descriptions of the operators $W^\bullet$ and T° involved in the magnetic symmetry are more awkward, becoming natural only as the smooth-spacetime limit is approached.[5,6] This article compares different ways to construct $W^\bullet$ and T° when the gauged group is $U(1)$, starting with the $U(1)$ version of the constructions that articles 40191 and 93302 described for any compact connected gauged group.

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[1] The constructions in this article work for any $d \geq 3$. The cases considered most often are $d = 3$ and $d = 4$.

[2] In quantum field theory, operators typically are typically not strictly localized on lower-dimensional manifolds of spacetime. In this article, *localized on X* means *localized in an arbitrarily small neighborhood of X*.

[3] Article 82508 describes $T^\bullet$ for an arbitrary compact gauged group.

[4] Article 89053 describes W° for an arbitrary compact gauged group.

[5] Article 40191 describes $W^\bullet$ for an arbitrary compact connected gauged group.

[6] Article 93302 describes T° for an arbitrary compact connected gauged group.

2 Notation and conventions

Let ϵ denote the characteristic distance between points of the lattice. In this article, taking the **smooth-spacetime limit** means adjusting the overall coefficient of the action to make the correlation length much larger than ϵ, using only states whose energy is much less than $1/\epsilon$, and considering only observables that preserve that condition. It doesn't mean a strict smooth-spacetime limit, which would be nice for aesthetic reasons but isn't necessary for physics.

In standard international units, the magnetic field B has dimensions mass/(time$\times$ charge), so magnetic flux has dimensions mass $\times$ length^2/(time $\times$ charge), as does $\hbar/q$ where q is a unit of electric charge.[7] The units convention in this article is similar to the units convention in article 44135 (which uses the one introduced in 26542), but with $\hbar = 1$.

Notation:

- d is the number of dimensions of spacetime.
- A manifold is called **closed** if it is compact and does not have a boundary.
- Γ is a closed $(d-3)$-dimensional submanifold of spacetime.
- $\tau(\Gamma)$ is a tubular neighborhood of Γ.
- Σ is a $(d-2)$-dimensional submanifold of spacetime with boundary $\partial\Sigma$.
- S denotes either the action (section 4) or a topological 2-sphere (section 6).
- $B(S)$ is the magnetic flux on S (section 6).
- D^k is a k-dimensional open ball.
- Spacetime is discretized as described in article 46333, which defines **links** and **plaquettes**. The discrete structure will be called a **lattice**.
- ℓ is an oriented link, and ℓ^{-1} is its counterpart with the opposite orientation.

[7] Article 00669

- $\square$ is an oriented plaquette, and $\square^{-1}$ is its counterpart with the opposite orientation.
- $u(\ell)$ is a **link variable**, a $U(1)$-valued variable associated with the link ℓ.
- $u(\square)$ is a **plaquette variable**, defined to be the product of the link variables around the perimeter of the oriented plaquette $\square$.
- $\theta(\square)$ is a lift of $u(\square)$ to the universal covering group $\mathbb{R}$ of $U(1)$.
- $z(\square)$ is a given element of $U(1)$ associated with the plaquette $\square$ (section 5).
- r is an irreducible unitary representation of the universal covering group $\mathbb{R}$ of $U(q)$ (section 6). The representation is specified by a real number β.
- The field[8] $\mathbb{R}$ of real numbers is, among other things, an abelian group with respect to addition. The same symbol $\mathbb{R}$ is also used to denote this abelian group without any further structure.

Notation for 't Hooft operators:

- Given a closed $(d-3)$-dimensional manifold Γ and an integer n, $T^{\circ}_{n}(\Gamma)$ denotes a type 1 't Hooft operator localized on Γ (section 5).[9]
- Given a $(d-2)$-dimensional surface Σ in spacetime and an element $z \in U(1)$, $T^{\bullet}_{z}(\Sigma)$ is the topological 't Hooft operator localized on Σ, as defined in article 82508.[10]

Article 22721 summarizes the differences between $T^{\bullet}$ and T°.

[8] This sentence uses the word *field* in the algebraic sense. Much of this article uses the word *field* with another one of its standard meanings, as in *magnetic field* and *electric field*.

[9] Article 93302 uses the notation $T^{\circ}_{\rho}(\Gamma)$, where ρ is the homomorphism from $U(1)$ to itself given by $g \mapsto g^{n}$.

[10] Article 44135 uses the notation $T^{\bullet}_{\delta\theta}(\Sigma)$ instead, with $z = e^{i\,\delta\theta} \in U(1)$, because it only considers the case $G = U(1)$.

3 Outline

- Section 4 reviews the model of a $U(1)$ gauge field using the path integral formulation in discrete spacetime.
- Section 5 reviews how 't Hooft operators T° are constructed in article 93302, specialized to the gauged group $U(1)$.
- Section 6 reviews how Wilson operators $W^\bullet$ are constructed in article 40191, specialized to the gauged group $U(1)$.
- Section 7 reviews the the model's magnetic symmetry, expressed in terms of the operators $W^\bullet$ and T°.
- Sections 8-10 describe the 't Hooft operators T° another way, without referring to a prescribed connection on a principal $U(1)$-bundle.
- When spacetime is 4-dimensional, the manifold Γ in $T^\circ(\Gamma)$ is 1-dimensional. Section 11 explains why Γ is sometimes described as the worldline of a magnetic monopole even though the model doesn't have any magnetically charged particles.
- Section 12 relates the magnetically charged 't Hooft operators T° to the topological 't Hooft operators $T^\bullet$ constructed in article 82508. This relationship is specific to the gauged group $U(1)$.
- Section 13 relates the path integral formulation used here to the single-time formulation used in article 44135.

4 Review of the model

The path integral has the form[11]

$$\Psi'[u]_{t'} \propto \int_{<t'} [du]\ e^{iS[u]} \Psi[u]_t \tag{1}$$

- Ψ and Ψ' are the initial and final states,
- $[u]_t$ denotes the set of link variables whose endpoints are both at time t,
- the integral is over of the link variables that have at least one endpoint in the range $\geq t$ and $< t'$ (with no more than one endpoint at time t'),
- each link variable is integrated over the gauged group $U(1)$.

The properties of the action $S[u]$ that will be important in this article include:

- It depends on link variables $u(\ell)$ only through plaquette variables $u(\square)$.[12]
- It is a sum of terms that each depends on only one plaquette variable:[13]

$$S[u] = \sum_{\square} c(\square) \left(1 - \frac{u(\square) + u^*(\square)}{2}\right) \tag{2}$$

  where $c(\square)$ are fixed complex-valued coefficients. Letting them be complex-valued allows for Wick rotation to a euclidean action.[11]
- Near the smooth-spacetime limit, the path integral is dominated by configurations of the link variables that come close to minimizing the euclidean action.[14]

[11] Article 89053

[12] This implies that $S[u]$ is invariant under gauge transformations.

[13] Section 2 defines the plaquette variables $u(\square)$.

[14] This can be achieved by taking the action's overall coefficient to be arbitrarily large. Whether the model remains nontrivial in the resulting smooth-spacetime limit depends on the gauged group G, the number of dimensions d, and the details of how the limit is taken (article 07611). In this article, allowing the model to become trivial is fine. Electrodynamics in smooth spacetime without matter is *trivial* in the sense that it doesn't have any interactions, but it's still interesting in other ways. The magnetic symmetry highlighted in this article is an example.

5 Magnetically charged 't Hooft operators

Article 93302 defined 't Hooft operators $T^\circ(\Gamma)$ for any compact connected gauged group G. This section reviews that construction in the special case $G = U(1)$, which is easier because the elements of $U(1)$ all commute with each other and plaquette variables are already gauge invariant without the help of a trace.

Including an 't Hooft operator T° in the integrand of the path integral means replacing

$$u(\square) + u^*(\square) \to z^*(\square)u(\square) + z(\square)u^*(\square) \tag{3}$$

in the action (2), where $z(\square)$ are specially chosen complex numbers. To choose the complex numbers $z(\square)$, think of the points, links, and plaquettes of the discrete spacetime as being embedded in smooth spacetime. Choose a closed oriented submanifold Γ of spacetime with codimension 3 that doesn't intersect any lattice points, links, or the surfaces of any plaquettes. The 't Hooft operator will be nominally localized on Γ. Suppose that Γ has a special type of neighborhood called a **trivial tubular neighborhood**. This is a neighborhood homeomorphic to $D^3 \times \Gamma$, where D^3 is a 3-ball.[15,16] Let $\tau(\Gamma)$ be such a neighborhood, chosen to be large compared to the lattice spacing in every direction transverse to Γ. Choose a principal $U(1)$-bundle over $\tau(\Gamma) \setminus \Gamma$ for which the flux on any 2-sphere[17] linked once with Γ is $2\pi n$ (with a sign depending on the orientations of Γ and the 2-sphere),[18] and choose a connection on this principal $U(1)$-bundle for which the field strength is well-distributed as described in article 49708.[19] Then $z(\square)$ is the holonomy around the perimeter of the plaquette $\square$ for each plaquette in τ. For plaquettes outside τ, set $z(\square) = 1$. The resulting operator will be denoted $T^\circ_n(\Gamma)$.

[15] Article 53600

[16] Every submanifold $\Gamma \subset M$ has a tubular neighborhood, but it may or may not be trivial (article 53600).

[17] This is a 2-sphere (surface of a 3d ball) in the topological sense, not necessarily shaped like a sphere.

[18] The condition that $\tau(\Gamma)$ is a *trivial* tubular neighborhood of Γ ensures that such a principal $U(1)$-bundle exists. The construction does not require the existence of such a principal $U(1)$-bundle everywhere on $M \setminus \Gamma$, but that does become important when considering the smooth-spacetime limit (section 7).

[19] The *well-distributed field strength* condition is the only part of the construction that depends on the spacetime metric.

6 Topological Wilson operators and magnetic flux

This section constructs the topological Wilson operators that generate the magnetic symmetry previewed in section 1.[20] In discrete spacetime, the construction is well-defined but unnatural. It becomes natural only in the smooth-spacetime limit.

If S is a closed oriented surface made of consistently-oriented plaquettes,[21] then the product of the plaquette variables on S

$$\prod_{\square \in S} u(\square) = 1 \tag{4}$$

because all the link variables in the product cancel each other in pairs.[22] This is a discrete version of the **Bianchi identity**.

Topological Wilson operators are defined by lifting the plaquette variables from $U(1)$ to the universal covering group of $U(1)$ before combining them over S. The result is not necessarily equal to 1. The rest of this section explains this in detail.

The gauged group $U(1)$ is compact, but its universal covering group is not.[23] Its universal covering group is isomorphic to the additive group $\mathbb{R}$ of real numbers.[24] The covering map $\sigma : \mathbb{R} \to U(1)$ is $\sigma(\theta) = e^{i\theta}$. For any real coefficient β, the map $r : \mathbb{R} \to \mathbb{C}$ defined by

$$r(\theta) = e^{i\beta\theta} \tag{5}$$

is a unitary irreducible representation of $\mathbb{R}$. If β is an integer, then this representation may be obtained from a representation $r'(e^{i\theta}) = e^{i\beta\theta}$ of $U(1)$ by composing it with the covering map:

$$r'\big(\sigma(\theta)\big) = e^{i\beta\theta}.$$

[20] This is a special case of the construction described in article 40191 for any compact connected gauged group.

[21] Section 2 defines the *orientation* of a plaquette.

[22] In more detail: let ℓ^{-1} denote the orientation-reversed version of the link ℓ and recall that link variables satisfy $u(\ell^{-1}) = (u(\ell))^{-1}$ (article 40191). When combined with the fact that on an oriented closed surface S, every link that occurs in a plaquette on S also occurs with the opposite orientation in an adjacent plaquette on S, this gives (4).

[23] Article 92035 defined **universal covering group**.

[24] This section represents the universal covering group using addition as the group operation, whereas article 40191 uses multiplication (which is standard when the group is not necessarily abelian).

If β is not an integer, then the representation (5) does not come from any representation of $U(1)$.[25]

In compact electrodynamics, the plaquette variables are $U(1)$-valued. The quantity $e^{i\theta} \in U(1)$ determines the value of θ only modulo 2π, but we can define the **canonical lift** $\tilde{\sigma}_1 : U(1) \to \mathbb{R}$ by $\tilde{\sigma}_1(e^{i\theta}) \equiv \theta$ where θ is the value closest to zero.[26] Applying $\tilde{\sigma}_1$ to a continuous $U(1)$-valued function f of spacetime would typically give a discontinuous $\mathbb{R}$-valued function of spacetime,[27] but $f \approx 1$ everywhere, then the resulting $\mathbb{R}$-valued function $\tilde{\sigma}_1(f)$ is continuous.

Near the smooth-spacetime limit, the path integral is dominated by configurations for which the values of the plaquette variables far from the core Γ of an 't Hooft operator $T^\circ(\Gamma)$ satisfy $u(\square) \approx 1$. If r is the representation (5) of $\mathbb{R}$ and S is a closed oriented surface made of oriented plaquettes,[28] we can define the **topological Wilson operator** $W_r^\bullet(S)$ by

$$W_r^\bullet(S) \equiv r\big(B(S)\big) = \exp\big(i\beta B(S)\big) \tag{6}$$

where $B(S)$ is the **magnetic flux**[29,30]

$$B(S) \equiv \sum_{\square \in S} \theta(\square)$$

with $\theta(\square) \equiv \tilde{\sigma}_1(u(\square))$.

[25] If β is not an integer, then $r' : e^{i\theta} \mapsto e^{i\beta\theta}$ is undefined.

[26] This is a special case of the canonical lift defined in article 60496 for any compact connected gauge group.

[27] This is why the construction is unnatural in discrete spacetime, becoming natural only in the smooth-spacetime limit.

[28] Section 2 defines the *orientation* of a plaquette.

[29] This article uses the system of units like the one in article 44135 but with $\hbar = 1$, so $B(S)$ is dimensionless.

[30] Article 51376 gives two definitions of magnetic flux, one of which is used in article 44135. Neither of those definitions relies on the concept of a lift from $U(1)$ to $\mathbb{R}$, but they both have other flaws as a result. Any definition of magnetic flux will be awkward in some way when spacetime is discrete, becoming natural only in the smooth-spacetime limit.

7 The magnetic symmetry

This section reviews the magnetic symmetry generated by the topological Wilson operators $W^\bullet(S)$ reviewed in section 6. The 't Hooft operators $T^\circ(\Gamma)$ reviewed in section 5 are charged with respect to this symmetry. This is the specialization to $G = U(1)$ of a result that article 40191 derives for any compact gauged group G.

The smooth-spacetime limit of the path integral is dominated by configurations of the gauge field that come close to minimizing the euclidean action. For a path integral with the modification (3), the smooth-spacetime limit is dominated by configurations with

$$u(\square) \approx z(\square). \tag{7}$$

When combined with the Bianchi identity (4) and the fact that the shifts $z(\square)$ are designed to vary only gradually as a function of location,[31] this ensures that $W_r^\bullet(S)$ gives the same result for every consistently-oriented 2-sphere S linked once with Γ.[32] The shifts $z(\square)$ were chosen so that the flux on such a 2-sphere is $2\pi n$ (with a sign depending on the orientation). If S is not linked with Γ, then the flux on S is zero, again because the principal $U(1)$-bundle used to construct the shifts $z(\square)$ has this property.

Using the definition (6) of $W_r^\bullet(S)$, those results may be expressed like this:

$$\tau\big(T_n^\circ(\Gamma),\ W_r^\bullet(S)\big) = e^{2\pi i\, nk\beta}\, \tau\big(T_n^\circ(\Gamma)\big)$$

where k is the linking number of the 2-sphere S with Γ and where $\tau(\cdots)$ denotes the path integral with the indicated operators inserted.[33,34] This is the **magnetic symmetry** of the model with a $U(1)$ gauge field.

[31] This condition may be violated very close to Γ, but it is satisfied at locations in the tubular neighborhood $\tau(\Gamma)$ that are far enough from Γ. In this section, S is understood to be a 2-sphere whose surface is far enough from Γ for the gradually-varying condition to hold.

[32] To deduce this, use (4) to infer that the flux must be an integer multiple of 2π, and then use the gradually-varying condition to infer that flux must be invariant under small changes in S, because this is the only way for a small change in the flux to preserve the integer-multiple-of-2π property (article 40191).

[33] Article 09181

[34] Elsewhere in this article, $\tau(\Gamma)$ denotes a tubular neighborhood of Γ (section 5 and footnote 31).

8 Another way to construct the shifts

Sections 9-10 will describe another way to construct such a configuration of shifts $z(\square)$ that section 5 used to define the 't Hooft operator $T^{\circ}(\Gamma)$.[35] Instead of referring to a principal $U(1)$-bundle, this approach uses a device that will be called *spokes*.

The bundle-based construction in section 5 and the spokes-based construction in sections 9-10 both have advantages. The bundle-based approach described in section 5 generalizes to any compact connected gauged group,[36] and it admits a natural definition of the *well-distributed field strength* condition.[37] The spokes approach has a different advantage: if the gauged group is $U(1)$ and Γ is the boundary of a submanifold Σ of spacetime with codimension 2, then the 't Hooft operators $T^{\circ}(\Gamma)$ described in this article may be constructed from the topological 't Hooft operators $T^{\bullet}(\Sigma)$ described in article 82508, and the spokes-based approach described in the following sections makes explaining that relationship relatively easy. This will be done in section 12.

Section 9 will describe the construction for $d = 3$, where the pattern is easy to visualize. Section 10 will explain how to generalize the construction to arbitrary $d \geq 3$.

[35] As in section 14, the construction works whenever Γ has a trivial tubular neighborhood.

[36] Article 93302

[37] Article 49708

9 Construction using spokes: the case $d = 3$

This section constructs a configuration of shifts $z(\square)$ when spacetime is three-dimensional ($d = 3$). The construction's details can be tuned to match the shifts $z(\square)$ that were used in section 5.

Let p be a point in Γ, and let $\tau(p)$ be a ball D^3 centered on p that is large compared to the discretization scale in all 3 dimensions. Choose a large number of **spokes** – line segments radiating outward from p to the boundary of $\tau(p)$ – adjusted so that they don't intersect any of the discrete spacetime's points or links. Figures 1, 2, and 3 illustrate the idea using a small number of spokes.

Let Ω be the set of oriented plaquettes that are intersected by these spokes and whose orientation is clockwise when viewed from p along a spoke.[38] Let K be the total number of spokes, and let $|k(\square)|$ be the number of spokes that intersect a given oriented plaquette $\square$. Choose the signs so that $k(\square) \geq 0$ and[39] $k(\square^{-1}) = -k(\square)$ for all $\square \in \Omega$. Then[40]

$$\sum_{\square \in S} \theta(\square) = 2\pi n \qquad \text{with } \theta(\square) \equiv \frac{2\pi n k(\square)}{K} \tag{8}$$

for every 2-sphere S made of oriented plaquettes in $\tau(\Gamma)$ that encloses p, with an appropriate orientation for S. By making K arbitrarily large, the spokes can be arranged to make the shifts $z(\square) = e^{i\theta(\square)}$ match those in section 5 with arbitrary precision. No matter how the spokes are arranged, equation (8) enforces the flux condition that was imposed in section 5.

[38] A choice of **orientation** for a plaquette is a choice of one of the two directions in which we could travel around its perimeter.

[39] $\square^{-1}$ denotes the oppositely-oriented version of $\square$.

[40] To deduce this, use the fact that S intersects all the spokes. The surface S could intersect a spoke more than once, but all but one of the intersections will occur in oppositely-oriented pairs, so the sum (8) is the same as it would be if S intersected each spoke exactly once. In that case, each of the K spokes contributes $2\pi n/K$ to the sum.

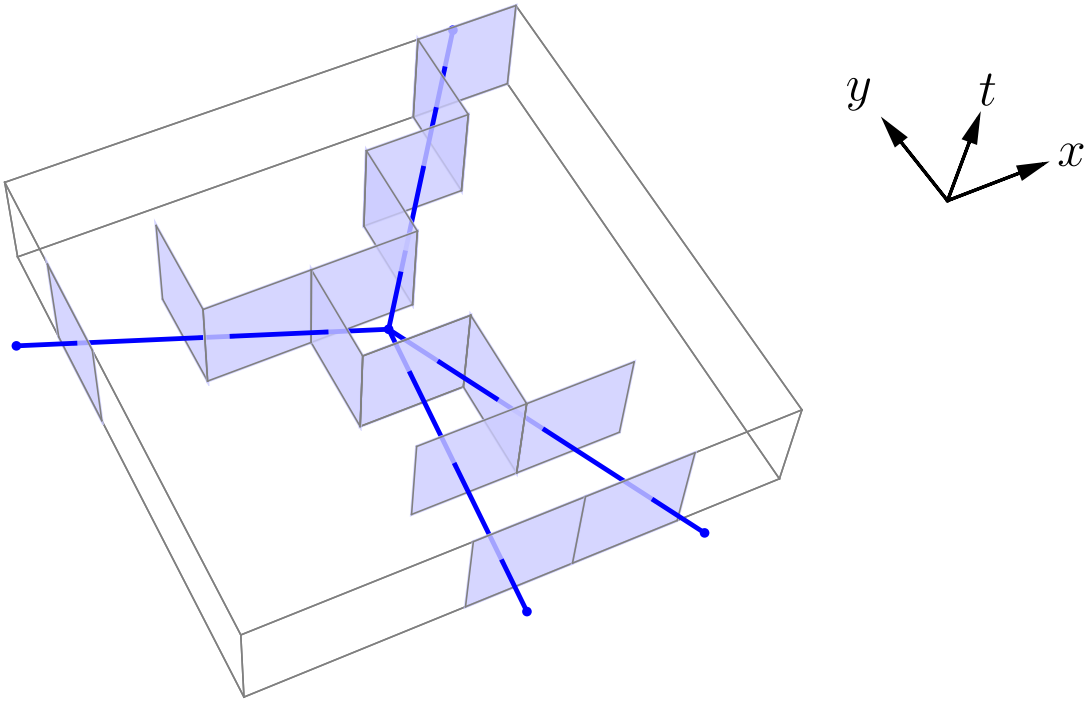

Figure 1 – Example of the construction in section 9, showing only four coplanar spokes instead of a large number of spokes distributed in three dimensions. The plaquettes intersected by the spokes are shaded. The boundary of the ball D^3 is just outside the perimeter of the large unshaded square.

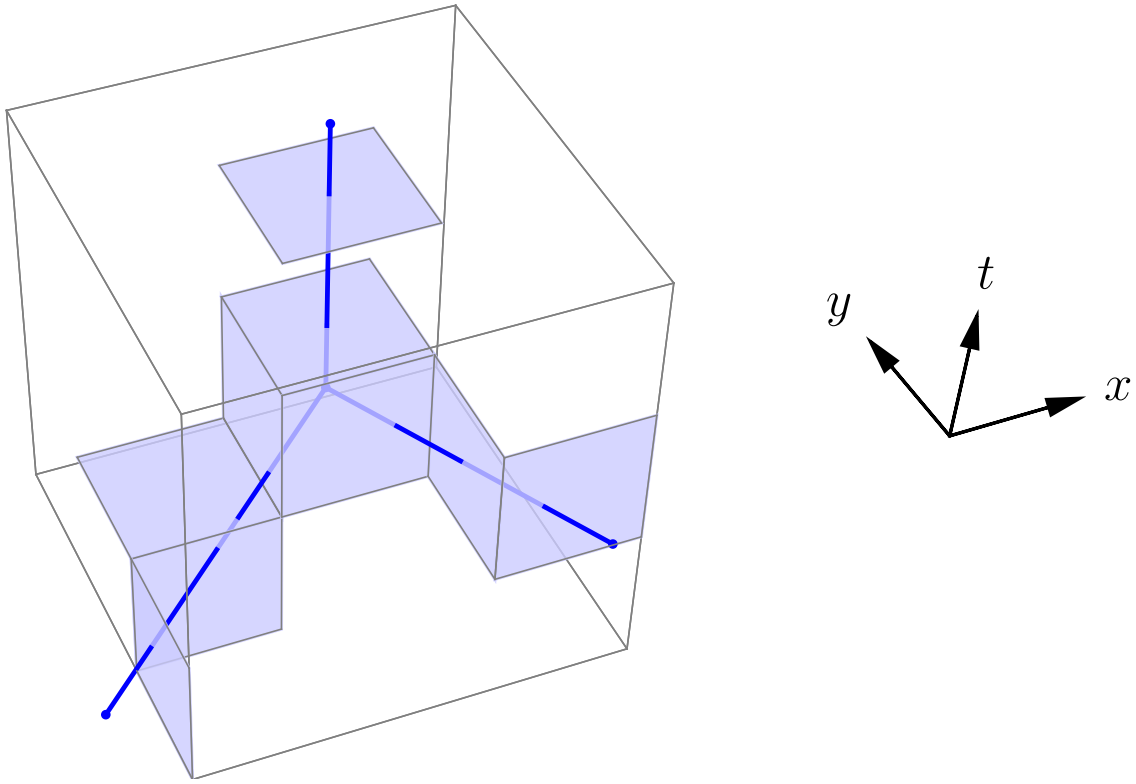

Figure 2 – Another example of the construction in section 9, like in figure 1 but with three non-coplanar spokes. The boundary of the ball D^3 is just outside the large unshaded cube.

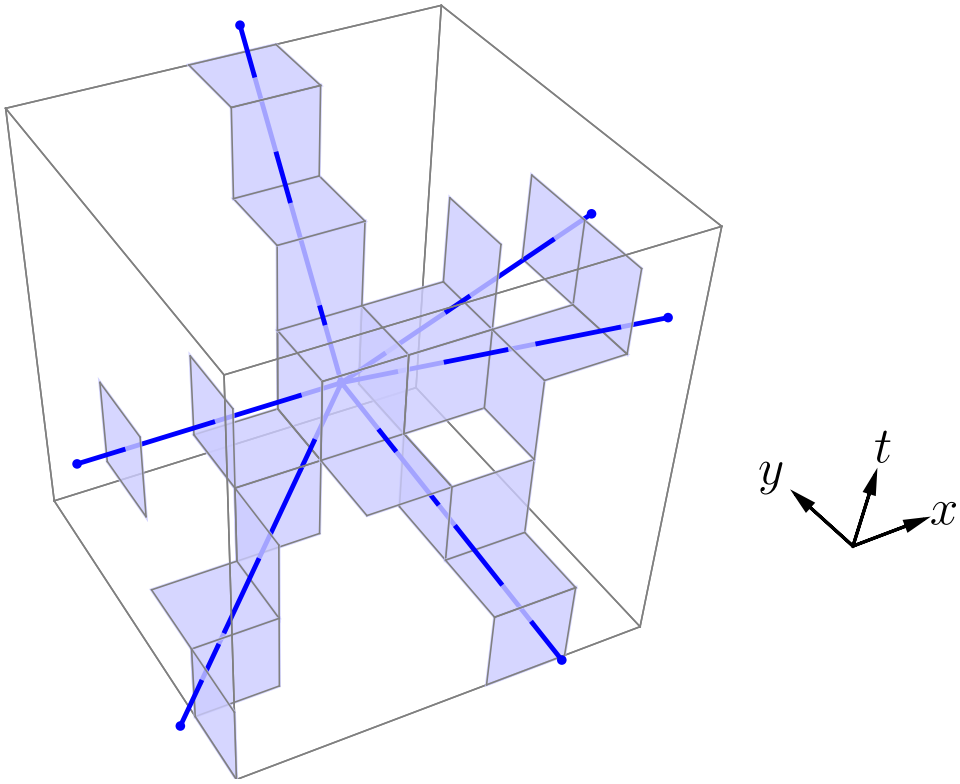

Figure 3 – Another example of the construction in section 9, like in figure 2 but with a few more spokes and a larger D^3. Each of the pictures in this series uses a relatively small D^3, only a handful of lattice spacings in width, because this makes the pictures easier to draw and easier to interpret. Ideally, the ball D^3 should be many lattice spacings in width (say, 10^9) and the number of spokes should be large enough to intersect every plaquette in D^3 at least once.

10 Construction using spokes for any $d \geq 3$

Section 9 constructed type 1 't Hooft operators in three-dimensional spacetime. This section generalizes the construction to d-dimensional spacetime for any $d \geq 3$.

First consider a 3-ball D^3. Choose a point p inside D^3 and let X be a configuration of one-dimensional spokes radiating from p to the boundary of D^3, like in section 9.[41] Now let Γ be a closed oriented smooth manifold with $d-3$ dimensions. Temporarily think of Γ as an abstract manifold by itself without the context of the ambient spacetime. We can form a topological space $X \times \Gamma$ that lives inside an abstract d-dimensional space $D^3 \times \Gamma$. The space $X \times \Gamma$ is a configuration of hypersurfaces that each have codimension 2 in $D^3 \times \Gamma$ and that all meet each other at a single copy of Γ (the higher-dimensional version of the original point p) that will be called the **hub**. These codimension 2 hypersurfaces will still be called **spokes**, even though each spoke is now $(d-2)$-dimensional instead of only 1-dimensional. None of the spokes intersect each other anywhere except at the hub.

Now suppose Γ has a neighborhood $\tau(\Gamma)$ homeomorphic to $D^3 \times \Gamma$.[42] This implies the existence of a homeomorphism from the abstract topological space $X \times \Gamma$ into $\tau(\Gamma)$. The spokes don't intersect each other in the abstract space $D^3 \times \Gamma$ except at the hub, and the homeomorphism into $\tau(\Gamma)$ preserves this property. Some of the spokes might intersect some points or links of the lattice, but since each spoke has codimension 2 in spacetime, we can eliminate all such intersections by deforming the spoke slightly.[43] For the same reason, if a spoke intersects a plaquette at all, then the intersection is generically a single point. The spoke may be deformed slightly if needed so that all intersections with plaquettes are single-point intersections. Choose the phases $\theta(\square)$ like in section 9. Any 2-sphere S that is contained in $\tau(\Gamma)$ and links once with Γ intersects each spoke in an odd number of points, after making slight adjustments if needed as explained above. If S is made of plaquettes, then this arrangement satisfies equation (8), just like in section 9.

[41] X would be a 1-dimensional manifold, but it's not a manifold at the point p where the spokes intersect. It is a kind of topological space called a 1-dimensional **CW complex** (article 93875).

[42] Section 5

[43] The spokes don't need to be straight.

11 Describing Γ as the worldline of a monopole

When spacetime is 4-dimensional ($d = 4$), an 't Hooft operator $T_n^\circ(\Gamma)$ is often described as the (not necessarily timelike) wordline of an external[44] magnetic monopole. This section explains the reason for that description.

When $d = 4$, the manifold Γ is 1-dimensional. Suppose Γ has a segment that is not orthogonal to the time direction, and consider a single point p in that segment. Let S be a closed surface that encloses the point p and that is made of plaquettes whose edges are all spacelike and situated at the same time as the point p.[45] When the operator $T_n^\circ(\Gamma)$ is present in the integrand of a path integral, equation (7) says that the smooth-spacetime limit of the path integral is dominated by configurations for which the flux on S is $2\pi n$. In smooth 3-dimensional space, any configuration with nonzero total magnetic flux on a 2-sphere is called a **magnetic monopole**.

The flux on S divided by the area of S gives the average strength of the electromagnetic field on S,[46] so by shrinking S toward the point p we can deduced that the electromagnetic field represented by those configurations must become singular at p when expressed in physically meaningful units (if $n \neq 0$). In classical electrodynamics, we could think of the point where the field becomes singular as the location of a pointlike particle, and that particle is also called a magnetic monopole. This is why Γ is sometimes described as the worldline of an external magnetic monopole. This language is still used even if the curve Γ is not timelike everywhere and even if the flux on a 2-sphere S linked with Γ involves both electric and magnetic components of the field.

[44] **External** means that it influences the quantum field(s) but is not influenced by the quantum field(s). The model used here does not have any magnetically charged particles.

[45] This implies that S is linked with Γ in spacetime.

[46] Article 51376 uses the plaquette variables $u(\square)$ to define a lattice version of the electromagnetic field operators.

12 Relating the two types of 't Hooft operator

This section relates the type of 't Hooft operator defined in section 8 to the type defined in article 82508 when the gauged group is $U(1)$.[47]

First, here's a review of the definition in article 82508, specialized to $U(1)$. For a given $z \in U(1)$ and a given plaquette $\square$, define an operator $T_z(\square)$ by

$$T_z(\square) \int [du]\ e^{iS[u]} \Psi[u] \equiv \int [du]\ e^{iS'[u]} \Psi[u] \tag{9}$$

where the modified action S' is obtained from the original action S by replacing $u(\square) \to zu(\square)$ and replacing $u(\square^{-1}) \to z^{-1}u(\square^{-1})$, like this:

$$S'\big(..., u(\square), u(\square^{-1}), ...\big) = S\big(..., zu(\square), z^{-1}u(\square^{-1}), ...\big).$$

All other plaquette variables (represented by "...") are unaffected. Now let Σ be a manifold with codimension 2 in spacetime that doesn't intersect any lattice points or links, doesn't intersect the surface of any plaquette at more than one point, and for which all such intersections are transverse (not tangential). Its boundary $\partial\Sigma$ should not intersect the surfaces of any plaquettes. The type of 't Hooft operator introduced in article 82508 is defined by

$$T_z^\bullet(\Sigma) = \prod_{\square \nearrow \Sigma} T_z(\square) \tag{10}$$

where the product is over oriented $\square$s that are intersect Σ transversely with positive orientation.[48] Those operators will be called **type 2 't Hooft operators**.[49] When $z = 1$, (10) reduces to the identity operator.

Choose a large integer $K \gg 1$ and a collection of manifolds $\Sigma_1, \Sigma_2, ..., \Sigma_K$ that are all equal to a designated manifold Σ except in a neighborhood $\tau(\partial\Sigma)$ of its

[47] Article 44135 explains how this works in the hamiltonian formulation.

[48] Article 82508 defines the orientation of an intersection.

[49] Article 22721 explains why.

boundary $\partial\Sigma$. Write this condition as

$$\Sigma_k = \Sigma_0 \cup \delta\Sigma_k$$

where Σ_0 is the part they all share and $\delta\Sigma_k$ is the unique part near $\partial\Sigma$. Each unique part $\delta\Sigma_k$ is contained inside $\tau(\partial\Sigma)$. The operator

$$\prod_k T_z^\bullet(\Sigma_k) = \prod_k \left(T_z^\bullet(\Sigma_0) T_z^\bullet(\delta\Sigma_k)\right) = T_{z^K}^\bullet(\Sigma_0) \prod_k T_z^\bullet(\delta\Sigma_k) \tag{11}$$

is **pre-smeared**[50] over $\tau(\partial\Sigma)$. If z is chosen so that $z^K = 1$, then using the identity $T_1^\bullet(\Sigma_0) = 1$ in (11) gives

$$\prod_k T_z^\bullet(\Sigma_k) = \prod_k T_z^\bullet(\delta\Sigma_k) \qquad \text{if } z^K = 1.$$

The effect this operator is localized in $\tau(\partial\Sigma)$. If we choose each $\delta\Sigma_k$ to be a *spoke* like section 10 described, then the result is a type 1 't Hooft operator:

$$\prod_k T_z^\bullet(\Sigma_k) = T_n^\circ(\partial\Sigma) \qquad \text{if } z = e^{2\pi i\, n/K}.$$

This relationship between the two types of 't Hooft operator holds when the gauged group G is $U(1)$, but it doesn't work for a group G whose center is finite. It doesn't work in that case because then elements of the center cannot be arbitrarily close to the identity operator as required by the spokes construction.

[50] Article 44135 explains the reason for this language. In that article, the smearing is only in the spatial dimensions. This article uses the path integral formulation, and here the smearing is in spacetime, not just in space.

13 Recovering the single-time formulation

Article 82508 introduces type 2 't Hooft operators $T^{\bullet}(\Sigma)$ in the path integral formulation for any compact gauged group G and explains how to relate this to the single-time formulation used in article 53519. When $G = U(1)$, the same relationship between the path integral and single-time formulations also works for type 1 't Hooft operators $T^{\circ}(\Gamma)$ if $\Gamma = \partial\Sigma$ and if we take all the manifolds Σ_k in section 12 to be situated at a single time.

14 References in this series

Article **00669** (https://cphysics.org/article/00669):
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Article **07611** (https://cphysics.org/article/07611):
"Asymptotic Freedom and the Continuum Limit of Yang-Mills Theory"

Article **09181** (https://cphysics.org/article/09181):
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