Rotational Motion in Higher-Dimensional Space

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Abstract This article derives the relationship between the angular momentum and angular velocity of a rigid body in D-dimensional space.

Contents

1 Review: momentum and angular momentum

As in article 33629, consider a system of pointlike objects with masses m_k whose locations $\mathbf{x}_k$ in D-dimensional space are governed by the equations of motion[1]

$$m_k\ddot{\mathbf{x}}_k = -\nabla_k V, \tag{1}$$

where V is a function of the locations $\mathbf{x}_k$. The index k runs from 1 to K, where K is the number of objects. Suppose that V is invariant under translations and rotations. The condition that V is invariant under translations can be written

$$\sum_k \nabla_k V = 0,$$

which implies that the total momentum

$$\mathbf{P}_{\text{total}} \equiv \sum_k m_k\dot{\mathbf{x}}_k \tag{2}$$

is conserved. The condition that V is invariant under rotations about any given time-independent center point $\mathbf{c}$ can be written

$$\sum_k (\mathbf{x}_k - \mathbf{c}) \wedge \nabla_k V = 0,$$

which implies that the total angular momentum about $\mathbf{c}$,

$$L_{\text{total}} \equiv \sum_k (\mathbf{x}_k - \mathbf{c}) \wedge (m_k\dot{\mathbf{x}}_k), \tag{3}$$

is conserved. The $\wedge$ notation was defined in article 33629. An equivalent definition, which will be more convenient in this article, is

$$\mathbf{a} \wedge \mathbf{b} \equiv \mathbf{a}\mathbf{b}^T - \mathbf{b}\mathbf{a}^T \tag{4}$$

where $\mathbf{a}$ is represented as a single-column matrix, and so is $\mathbf{b}$. In this representation, $\mathbf{a} \wedge \mathbf{b}$ is an antisymmetric $D \times D$ matrix. It is nonzero only if $\mathbf{a}$ and $\mathbf{b}$ are linearly independent.

[1] Each overhead dot is a derivative with respect to time.

2 Describing the motion of a rigid body

For the rest of this article, suppose that the distance between each pair of objects is fixed,[2] so that the system describes a single rigid object whose total mass is distributed among K points $\mathbf{x}_k$, with m_k the mass associated with point $\mathbf{x}_k$. Our goal is to write the conserved quantities (2) and (3) in a way that is more convenient for studying the motion of such a a rigid object.

In D-dimensional space, a rotation is described by a $D \times D$ matrix R satisfying

$$RR^T = R^T R = 1 \qquad\qquad \det R = 1. \tag{5}$$

The "1" in the first equation is the identity matrix, the superscript T means transpose, and "det" means determinant. Define the **center of mass**:

$$\mathbf{x} = \sum_k \frac{m_k \mathbf{x}_k}{m} \qquad\qquad \text{with} \quad m \equiv \sum_k m_k. \tag{6}$$

The assumption that the body is **rigid** means that we can write

$$\mathbf{x}_k = R\mathbf{b}_k + \mathbf{x} \tag{7}$$

where R is a time-dependent rotation matrix and $\mathbf{b}_k$ is the initial location of the kth mass relative to the center of mass. Equations (6)-(7) imply

$$\sum_k m_k \mathbf{b}_k = 0. \tag{8}$$

By definition, $\mathbf{b}_k$ is independent of time, so all of $\mathbf{x}_k$'s time-dependence comes from R and $\mathbf{x}$.

[2] In a model of the form (1), this can be achieved by choosing V to be have an enormous magnitude whenever any of the distances deviates significantly from its nominal value, but we won't need to do that here. In this article, equation (1) serves only to motivate the idea that (2) and (3) are conserved. The rest of this article treats those conservation laws as axioms.

3 Angular momentum and angular velocity

Use equation (7) to see that the total angular momentum (3) is

$$L_{\text{total}} = \sum_k m_k(R\mathbf{b}_k + \mathbf{x} - \mathbf{c}) \wedge (\dot{R}\mathbf{b}_k + \dot{\mathbf{x}}),$$

and then use (8) to reduce this to

$$L_{\text{total}} = L + (\mathbf{x} - \mathbf{c}) \wedge \mathbf{P}_{\text{total}}$$

with

$$L \equiv \sum_k m_k(R\mathbf{b}_k) \wedge (\dot{R}\mathbf{b}_k). \tag{9}$$

The quantity L is the angular momentum of the body about its own center of mass, and the remainder $(\mathbf{x} - \mathbf{c}) \wedge \mathbf{P}_{\text{total}}$ is the angular momentum of the center of mass about the arbitrary point $\mathbf{c}$.

Take the time-derivative of the identity $RR^T = 1$ to see the matrix $W \equiv -\dot{R}R^T$ is antisymmetric, and then use $R^T R = 1$ to get $\dot{R} = -WR$. The matrix W is called the **angular velocity**. Like the angular momentum L, the angular velocity is an antisymmetric matrix.[3] Use equations (4) and (9) to see that the angular momentum L can be written in terms of the angular velocity W like this:

$$L = MW + WM \tag{10}$$

where M is the symmetric matrix

$$M \equiv RM_0R^T \qquad\qquad M_0 \equiv \sum_k m_k \mathbf{b}_k \mathbf{b}_k^T. \tag{11}$$

The matrix M (or its initial value M_0) describes the object's **rotational inertia**, just like its total mass $\sum_k m_k$ describes is **linear inertia**. Section 5 explains how M is related to the traditional moment of inertia tensor when $D = 3$.

[3] When $D = 3$, a $D \times D$ antisymmetric matrix happens to have 3 independent components, which the traditional formulation arranges into a "vector" – a trick that only works for $D = 3$.

4 Kinetic energy

The total kinetic energy is

$$E = \frac{1}{2}\sum_k m_k \dot{\mathbf{x}}_k^2.$$

Use (7) and (8) to get

$$E = \frac{1}{2}\text{Trace}(\dot{R}M_0\dot{R}^T) + \frac{1}{2}m\dot{\mathbf{x}}^2.$$

Use the definitions of W and M to see that this may also be written

$$E = \frac{1}{2}\text{Trace}(WMW^T) + \frac{1}{2}m\dot{\mathbf{x}}^2, \tag{12}$$

and use (10) and $\dot{R}R^T + R\dot{R}^T = 0$ to see that it may also be written

$$E = \frac{1}{4}\text{Trace}(W^T L) + \frac{1}{2}m\dot{\mathbf{x}}^2. \tag{13}$$

This shows that the kinetic energy is the sum of two terms: the first term is the part due to the object's rotational motion about its center of mass, and the second term is part due to the motion of the center of mass.

5 The traditional formulation in 3d space, part 1

The formulation described in the preceding sections works in D-dimensional space for any D. In that formulation, the angular momentum L and angular velocity W are represented by antisymmetric $D \times D$ matrices. Such a matrix has

$$\frac{(D-1)D}{2}$$

independent components, which is the number of components above the diagonal.[4] When $D = 3$, the number of independent components is 3. The traditional formulation uses this coincidence to treat the angular momentum and angular velocity as though they were vectors, even though they're really not.[5] To relate the preceding formulation to the traditional one, write the components of L and W as

$$L = \begin{bmatrix} 0 & L_3 & -L_2 \\ -L_3 & 0 & L_1 \\ L_2 & -L_1 & 0 \end{bmatrix} \qquad W = \begin{bmatrix} 0 & W_3 & -W_2 \\ -W_3 & 0 & W_1 \\ W_2 & -W_1 & 0 \end{bmatrix} \tag{14}$$

and write the components of the rotational inertia matrix M as

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{12} & M_{22} & M_{23} \\ M_{13} & M_{23} & M_{33} \end{bmatrix}.$$

Then equation (10) is equivalent to

$$L_j = \sum_k I_{jk} W_k \qquad I_{jk} \equiv \text{Trace}(M)\delta_{jk} - M_{jk}.$$

The matrix I_{jk} is the traditional **moment of inertia tensor**.

[4] The components below the diagonal are the negatives of these, and the diagonal components are zero.

[5] Traditional texts acknowledge this by calling them **axial vectors** or **pseudovectors** (section 6)

6 The traditional formulation in 3d space, part 2

The traditional formulation in three-dimensional space uses a construct called the **cross product**. Given two vectors $\mathbf{a}$ and $\mathbf{b}$, their cross product

$$\mathbf{c} \equiv \mathbf{a} \times \mathbf{b} \tag{15}$$

is an axial vector. It's called an **axial** vector because it doesn't transform like a vector under reflections, but it does transform like a vector under rotations. In matrix notation, if $\mathbf{a} \to R\mathbf{a}$ and $\mathbf{b} \to R\mathbf{b}$, then $\mathbf{c} \to R\mathbf{c}$. In other words, equation (15) implies

$$R\mathbf{c} = (R\mathbf{a}) \times (R\mathbf{b}) \tag{16}$$

for all rotations R. This works even though the left-hand side has only one factor of R and the right-hand side has two factors of R. The proof uses $\det R = 1$ (equation (5)).

As explained in article 81674, this has a generalization to any number of dimensions D, but using the wedge product in place of the cross product. For general D, the wedge product of *two* vectors doesn't have the same number of components as a vector, but the wedge product of $D-1$ vectors does. The wedge product of $D-1$ vectors transforms under rotations just like a single vector does, again because $\det R = 1$.[6] The result (16) becomes a special case of this after the components of $\mathbf{a} \wedge \mathbf{b} = \mathbf{a}\mathbf{b}^T - \mathbf{b}\mathbf{a}^T$ are arranged into an (axial) vector as shown in equations (14).

In contrast, quantities like angular momentum, angular velocity, and torque do *not* transform like vectors when $D \neq 3$. They obviously can't, because they don't even have the same number of components as a vector when $D \neq 3$. Those quantities are always **bivectors**: they transform under rotations like $\mathbf{a} \wedge \mathbf{b}$ does, for two vectors $\mathbf{a}, \mathbf{b}$, with *two* factors of R, not with $D-1$ factors of R. These are the same thing only in the special case $D = 3$.

[6] This becomes obvious when the determinant is defined using the wedge product, as explained in article 81674.

7 References in this series

Article **33629** (https://cphysics.org/article/33629):
"Conservation Laws and a Preview of the Action Principle" (version 2022-02-05)

Article **81674** (https://cphysics.org/article/81674):
"Can the Cross Product be Generalized to Higher-Dimensional Space?" (version 2022-02-06)