Conservation Laws from Noether's Theorem

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Abstract This article uses the lagrangian formalism that is introduced in article 46044 to give another perspective on the conservation laws that are introduced in article 33629.

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1 Introduction

Article 33629 introduced a few conservation laws for a system of K pointlike objects governed by the equations of motion

$$m_k\ddot{\mathbf{x}}_k(t) = -\nabla_k V\big(\mathbf{x}_1(t),\, \mathbf{x}_2(t),\, ...\big), \tag{1}$$

with $k \in \{1, 2, ..., K\}$. The function $\mathbf{x}_k(t)$ describes the behavior of the kth object, and the equations of motion (1) tell us which behaviors are physically allowed.[1] Article 46044 showed how the equations of motion can be expresed in terms of a lagrangian

$$L = \frac{1}{2}\left(\sum_k m_k\dot{\mathbf{x}}_k^2(t)\right) - V\big(\mathbf{x}(t)\big), \tag{2}$$

where $V\big(\mathbf{x}(t)\big)$ is an abbreviation for

$$V\big(\mathbf{x}(t)\big) \equiv V\big(\mathbf{x}_1(t), \mathbf{x}_2(t), ..., \mathbf{x}_K(t)\big).$$

Explicitly, the equations of motion (1) are

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\mathbf{x}}_k(t)} = \frac{\partial L}{\partial\mathbf{x}_k(t)}. \tag{3}$$

This article combines those two themes by showing how conservation laws can be expressed in terms the lagrangian. This leads to a general connection between conservation laws and symmetries. The general connection is called **Noether's theorem**, also spelled **Nöther's theorem**.

[1] That is, which behaviors are physically allowed *according to this model*. This model is not a perfect representation of the real world, but it's good enough for some purposes.

2 Setup for Noether's theorem

For any finite time interval I, define the action

$$S_I = \int_I dt\ L\big(\mathbf{x}_1(t),\, \mathbf{x}_2(t),\, ...,\, \dot{\mathbf{x}}_1(t)\, \dot{\mathbf{x}}_2(t)\, ...\big). \tag{4}$$

The lagrangian

$$L(\mathbf{x}_1,\, \mathbf{x}_2,\, ...,\, \mathbf{y}_1,\, \mathbf{y}_2,\, ...) \tag{5}$$

is an ordinary function of the indicated variables. Equation (2) shows an example in which the function (5) is $\frac{1}{2}\left(\sum_k m_k \mathbf{y}_k^2\right) - V(\mathbf{x})$.

Under any infinitesimal[2] variation of the behavior $\mathbf{x}_k(t)$, whether or not the bevavior satisfies the equations of motion, the resulting variation of (4) is

$$\delta S_I = \int_I dt\ \sum_k \left(\frac{\partial L}{\partial \mathbf{x}_k(t)} \cdot \delta\mathbf{x}_k(t) + \frac{\partial L}{\partial \dot{\mathbf{x}}_k(t)} \cdot \delta\dot{\mathbf{x}}_k(t)\right). \tag{6}$$

The partial derivatives of L are evaluated as though $\mathbf{x}_k$ and $\dot{\mathbf{x}}_k$ were independent variables, because L itself allows them to be specified independently. Only *after* evaluating these partial derivatives do we take $\dot{\mathbf{x}}_k$ to be the time-derivative of $\mathbf{x}_k$.

We could define a symmetry to be any variation $\delta\mathbf{x}_k$ that gives $\delta S_I = 0$ for all behaviors $\mathbf{x}_k$, but that definition would exclude time-translation symmetry (because the interval I is finite). We can remove that limitation by defining a **symmetry** to be any variation $\delta\mathbf{x}_k$ for which δS_I can be written

$$\delta S_I = \int_I dt\ \frac{d}{dt}\Lambda(t) \tag{7}$$

for all behaviors $\mathbf{x}_k$, where $\Lambda(t)$ depends only on the $\mathbf{x}_k(t)$s and $\dot{\mathbf{x}}_k(t)$s at time t. Thanks to the fundamental theorem of calculus, the right-hand side of (7) does not depend on the system's behavior except at the endpoints of the interval I, which can be pushed arbitrarily far into the future/past.

[2] Infinitesimal means that we can neglect higher-order terms like $(\delta\mathbf{x}_k)^2$.

3 Noether's theorem

Now, suppose that the behavior $\mathbf{x}_k(t)$ satisfies the equations of motion (3) and *also* suppose that $\delta\mathbf{x}_k$ is a symmetry (7). Combining equations (3), (6), and (7) gives

$$\int_I dt\ \frac{d}{dt}\left[\sum_n \frac{\delta L}{\delta\dot{\mathbf{x}}_n}\cdot\delta\mathbf{x}_n - \Lambda\right] = 0. \tag{8}$$

The action principle requires this to hold for all time intervals I, so it implies

$$\frac{d}{dt}\left[\sum_n \frac{\delta L}{\delta\dot{\mathbf{x}}_n}\cdot\delta\mathbf{x}_n - \Lambda\right] = 0. \tag{9}$$

This is a conservation law. It holds whenever both of these conditions are satisfied: the behavior $\mathbf{x}_k(t)$ satisfies the equations of motion (3), and the variation $\delta\mathbf{x}_k$ is a symmetry (7).

Some familiar examples are shown in the following sections.

4 Example: conservation of momentum

Suppose that L depends on the behaviors $\mathbf{x}_k(t)$ only through their time-derivatives $\mathbf{x}_k$ and differences $\mathbf{x}_j - \mathbf{x}_k$. Then L is invariant under translations in space, which shift all of the $\mathbf{x}_k(t)$ by the same amount $\mathbf{c}$. Therefore, the variation $\delta\mathbf{x}_k = \mathbf{c}$ satisfies the definition of symmetry (7) with $\Lambda = 0$. When specialized to this symmetry, Noether's theorem (9) reduces to the conservation law

$$\frac{d}{dt}\sum_k \frac{\delta L}{\delta \dot{\mathbf{x}}_k}\cdot \mathbf{c} = 0.$$

This holds for all (time-independent) $\mathbf{c}$, so we can write it more simply as

$$\frac{d}{dt}\sum_k \frac{\delta L}{\delta \dot{\mathbf{x}}_k} = 0.$$

This says that the quantity

$$\sum_k \frac{\delta L}{\delta \dot{\mathbf{x}}_k} \tag{10}$$

is conserved whenever L is invariant under translations in space. (**Conserved** means that its time-derivative is zero if the behaviors $\mathbf{x}_k(t)$ satisfy the equations of motion.) This conserved quantity is called the system's **momentum**. Momentum is the name of the conserved quantity that is associated with spatial translation symmetry.

In the special case (2), the quantity (10) is

$$\sum_k m_k \dot{\mathbf{x}}_k, \tag{11}$$

which is the familiar expression for the total momentum of a system of pointlike objects that are governed by equations of motion of the form (1).

5 Example: conservation of energy

Suppose that the action is invariant under translations in time, which means that L does not depend on time except through its inputs $\mathbf{x}_k(t)$ and $\dot{\mathbf{x}}_k(t)$. Then

$$\delta\mathbf{x}_k = \dot{\mathbf{x}}_k\,\delta t \qquad\qquad \text{for all } k$$

is a symmetry (7) with[3]

$$\frac{d\Lambda}{dt} = \frac{dL}{dt}\delta t.$$

When specialized to this symmetry, Noether's theorem (9) reduces to the conservation law

$$\frac{d}{dt}\left[\sum_n \frac{\delta L}{\delta\dot{\mathbf{x}}_n}\cdot\dot{\mathbf{x}}_k - L\right] = 0.$$

This says that the quantity

$$\sum_n \frac{\delta L}{\delta\dot{\mathbf{x}}_n}\cdot\dot{\mathbf{x}}_k - L \tag{12}$$

is conserved whenever L is invariant under translations in time. This conserved quantity is called the system's **energy**. Energy is the name of the conserved quantity that is associated with time-translation symmetry.

In the special case (2), the quantity (12) is

$$\frac{1}{2}\sum_k m_k\dot{\mathbf{x}}_k^2 + V, \tag{13}$$

which is the familiar expression for the total energy of a system of pointlike objects that are governed by equations of motion of the form (1).

[3] δt is an infinitesimal constant. It does not depend on time.

6 Example: conservation of angular momentum

Suppose that L depends on the behaviors $\mathbf{x}_k(t)$ only through the magnitudes $|\dot{\mathbf{x}}_k|$ of the velocities and through the relative distances $|\mathbf{x}_j - \mathbf{x}_k|$. Then L is invariant under rotations. The effect of an infinitesimal rotation about the origin is

$$\delta\mathbf{x}_k = B\mathbf{x}_k\,\delta\theta$$

where B is an antisymmetric matrix[4] independent of t and k and $\delta\theta$ is an infinitesimal angle. This satisfies the definition of symmetry (7) with $\Lambda = 0$. To prove this, use the fact that B is antisymmetric implies $\mathbf{x}\cdot(B\mathbf{x}) = 0$ for any $\mathbf{x}$. When specialized to this symmetry, Noether's theorem (9) gives the conservation law

$$\frac{d}{dt}\left(\sum_k \frac{\delta L}{\delta\dot{\mathbf{x}}_k}\cdot B\mathbf{x}_k\,\delta\theta\right) = 0.$$

This says that the quantity

$$\sum_k \frac{\delta L}{\delta\dot{\mathbf{x}}_k}\cdot B\mathbf{x}_k \tag{14}$$

is conserved whenever L is invariant under rotations. This conserved quantity is called the system's **angular momentum**, specifically its angular momentum about the origin in the plane(s) defined by B. Angular momentum is the name of the conserved quantity that is associated with rotational symmetry.

In the special case (2), the quantity (14) is

$$\sum_k m_k\dot{\mathbf{x}}_k\cdot B\mathbf{x}_k \tag{15}$$

which is almost the familiar expression for the total angular momentum of a system of pointlike objects, except that it's written in terms of B instead of using a cross-product. The cross-product is defined only in three-dimensional space, but (15) makes sense in any number of dimensions (articles 12707 and 81674).

[4] In this equation, $\mathbf{x}_k$ is represented as a single-column matrix, and $B\mathbf{x}_k$ is its product with a square matrix B.

7 Example: the relativity of velocity

Consider a lagrangian of the form

$$L = \frac{1}{2}\sum_k m_k \dot{\mathbf{x}}_k^2 - V \tag{16}$$

where V depends only on the $\mathbf{x}_k$s, not on their time-derivatives. The equations of motion (3) for such a model have the form (1). Suppose that V is invariant under translations in space, and suppose also that V is independent of time except through the time-dependence of its arguments $\mathbf{x}_k(t)$. Then V is also invariant under $\delta\mathbf{x}_k = \mathbf{c}(t)$, where $\mathbf{c}(t)$ is any time-dependent quantity that is the same for all k. In particular, V is invariant under

$$\delta\mathbf{x}_k(t) = \mathbf{c}t \tag{17}$$

for any constant $\mathbf{c}$. This satisfies the definition of symmetry (7) with[5]

$$\Lambda = \sum_k m_k \mathbf{x}_k \cdot \mathbf{c}.$$

When specialized to this symmetry, Noether's theorem (9) says that the quantity

$$\sum_k \frac{\delta L}{\delta \dot{\mathbf{x}}_k} t - \sum_k m_k \mathbf{x}_k$$

is conserved when L is given by (16). This conserved quantity can also be written

$$\sum_k m_k(\dot{\mathbf{x}}_k t - \mathbf{x}_k).$$

This conserved quantity does not have a special name. Article 33629 mentions its significance.

[5] Remember that terms of order $\mathbf{c}^2$ are ignored (footnote 2).

8 References in this series

Article **12707** (https://cphysics.org/article/12707):
"Rotational Motion in Higher-Dimensional Space" (version 2022-02-05)

Article **33629** (https://cphysics.org/article/33629):
"Conservation Laws and a Preview of the Action Principle" (version 2022-02-05)

Article **46044** (https://cphysics.org/article/46044):
"The Action Principle in Newtonian Physics" (version 2022-02-05)

Article **81674** (https://cphysics.org/article/81674):
"Can the Cross Product be Generalized to Higher-Dimensional Space?" (version 2022-02-06)