Classical Scalar Fields in Curved Spacetime

Randy S

Abstract This article introduces classical field theory in a curved spacetime background, using a scalar field as an example. The use of a general background metric enables defining the **Hilbert stress-energy tensor**. The **canonical stress-energy tensor** that was introduced in in article 49705 is associated with the translation symmetry of flat spacetime via Noether's first theorem, whereas the Hilbert stress-energy tensor arises naturally from the equation of motion for the metric field in general relativity. Article 32191 explains why these two differently-defined stress-energy tensors are consistent with each other.

Contents

1 Two stress-energy tensors: canonical and Hilbert

Article 49705 introduced what I'll call the **canonical stress-energy tensor**, a collection of conserved currents associated with spacetime translation symmetry via Noether's (first) theorem. For a system of fields with components ϕ_n, the canonical stress-energy tensor is

$$T^{ab}(x) = \sum_n \frac{\delta L(x)}{\delta\, \partial_a \phi_n(x)} \partial^b \phi_n(x) - g^{ab} L(x), \tag{1}$$

where L is the lagrangian[1] and g^{ab} are the components of the (inverse) spacetime metric, which must be independent of the coordinates in a model with translation symmetry. This article introduces the **Hilbert stress-energy tensor**

$$\boxed{T^{ab}(x) \equiv \frac{-2}{\sqrt{|\det g(x)|}} \frac{\delta S}{\delta g_{ab}(x)},} \tag{2}$$

where S is the action and $|\det g|$ is the magnitude of the determinant of the metric. This definition works naturally even in curved spacetime, where translation symmetry is absent. This article highlights a different kind of symmetry that leads to a **covariant conservation law** for (2) even when spacetime is curved. It reduces to the usual local conservation law when spacetime is flat.

Even though they're both called *stress-energy* tensors,[2] their definitions are different. When spacetime is flat, the canonical and Hilbert stress-energy tensors can both be defined, but they may or may not be equal to each other, depending on the model.[3] The Hilbert stress-energy tensor has nicer properties:[4] it is automatically symmetric ($T^{ab} = T^{ba}$), and it is automatically invariant under gauge transformations in the context of electrodynamics.

[1] This expression for T^{ab} assumes that L depends only on the fields and their first derivatives.

[2] Here, the word *tensor* is being used as an abbreviation for *tensor field* (article 09894).

[3] They can often be made equal to each other by exploiting the non-uniqueness highlighted in the next section.

[4] The stress-energy tensor defined by (1) also has these properties in the special case studied in article 49705, but not in some other models like electrodynamics.

2 Non-uniqueness of the stress-energy tensors

The canonical and Hilbert stress-energy tensors are both non-unique:

- In a model with translation symmetry, Noether's theorem gives $\partial_a T^{ab} = 0$ whenever the fields satisfy their equations of motion. But Noether's theorem only gives an expression for $\partial_a T^{ab}$, not for T^{ab} itself, so the canonical stress-energy tensor (1) is not unique.[5]

- To define the Hilbert stress-energy tensor, we need to specify how the model depends on the spacetime metric. Even if we only care about the flat-spacetime case, we still need to temporarily generalize the model to arbitrary not-quite-flat metrics. This can be done in more than one way, and the output of the definition (2) depends on which generalization we choose. Section 7 shows that this dependence can persist even after specializing the result to flat spacetime.

In this article, the metric field is a prescribed background field. This means that the metric field is not influenced by anything else in the model: it is exempt from the action principle. I like to call this **generalized special relativity** (article 33547).[6] Contrast this to **general relativity**, where the metric field is subject to the action principle: it both influences and is influenced by the model's other dynamic entities. In general relativity, the arbitrary choice that would have made T^{ab} non-unique is no longer a choice at all, because the metric field's behavior is governed by the model's own equations of motion instead of being prescribed.

[5] Noether's theorem associates translation symmetry with an expression of the form $\partial_a T^{ab}$ that is zero whenever the fields satisfy their equations of motion. If K^{ab} is such that $\partial_a K^{ab}$ is *identically* zero (without using the equations of motion), then $T^{ab} + K^{ab}$ is another conserved quantity that is still associated with translation symmetry, because Noether's theorem can't tell the difference between T^{ab} and $T^{ab} + K^{ab}$.

[6] This name is not standard. It doesn't have a standard name, except maybe "physics in curved spacetime." Sometimes it's even called "general relativity," even though this name is also used for the case where the metric field is subject to the action principle.

3 Scalar field in curved spacetime

Let $g_{ab}(x)$ be the components of a prescribed metric field with lorentzian signature, and let N be the number of spacetime dimensions (normally $N = 4$). Consider the model of a single scalar field ϕ with action

$$S = \int d^N x \sqrt{|\det g|}\, L \qquad\qquad L \equiv \frac{g^{ab}(\partial_a\phi)(\partial_b\phi)}{2} - V(\phi), \tag{3}$$

where g_{ab} is a prescribed background metric field. When the metric is flat, (3) reduces to the action that was used in article 49705.[7]

We can better appreciate the structure of (3) by starting with the coordinate-free definitions of tensor fields reviewed in article 09894. The quantities $\partial_a\phi$ are the components of the **differential** of the scalar field ϕ. In a coordinate representation, the differential of ϕ is

$$dx^a\,\partial_a\phi. \tag{4}$$

This is (the coordinate representation of) a **one-form** field. The combination

$$g^{ab}(\partial_a\phi)(\partial_b\phi)$$

is (the coordinate representation of) the scalar field formed from the (inverse) metric field and two copies of the one-form field (4). The quantity $V(\phi)$ is also a scalar field, which is clear because it doesn't involve any derivatives. The integration measure $d^N x$ changes when the coordinate system is changed, but the combination $d^N x \sqrt{|\det g|}$ does not, so an integral of the form $\int d^N x \sqrt{|\det g|}\, L(x)$ has this same form in *every* coordinate system if L is a scalar field. In particular, (3) has the same form in every coordinate system. This ensures that the stress-energy tensor obtained from the definition (2) has a natural coordinate-free meaning as a tensor field.

[7] Equation (3) is not the only way to generalize the flat-spacetime version to curved spacetime.

4 The equation of motion

In this model, the influence is assumed to be only one-way: the metric field influences the scalar field, but not conversely. This means that the model's equations of motion include

$$\frac{\delta S}{\delta \phi} = 0 \tag{5}$$

but not $\delta S/\delta g_{ab} = 0$, which would be the equation of motion for the metric field if the influence went both ways (as it does in general relativity). For the action (3), equation (5) is equivalent to the Euler-Lagrange equations (article 49705)

$$\partial_a \frac{\delta \hat{L}}{\delta\, \partial_a \phi} = \frac{\delta \hat{L}}{\delta \phi} \tag{6}$$

with[8]

$$\hat{L} \equiv \sqrt{|\det g|}\, L. \tag{7}$$

Use (7) in (6) to get the explicit equation of motion

$$\partial_a\Big(\sqrt{|\det g|}\, g^{ab}\partial_b\phi\Big) + \sqrt{|\det g|}\, V'(\phi) = 0. \tag{8}$$

This equation governs the behavior of the scalar field in this model.

The way equation (8) depends on the metric g might have been hard to guess if we had tried to directly generalize the flat-spacetime equation of motion. We might have tried a derivative term like $g^{ab}\partial_a\partial_b\phi$ instead, but this combination does not correspond to a tensor field. Directly generalizing the equation of motion is easier using the concept of a **covariant derivative** ∇_a. This will be done in section 6.

[8] In Forger and Römer (2003), on pages 20 and 43, the quantity L in equation (3) is called the **lagrangian function**, the quantity $\hat{L}$ defined in (7) is called the **lagrangian density**, and $\hat{L}\, d^N x$ is called the **lagrangian**.

5 Calculation of the stress-energy tensor

To evaluate the definition (2), we need these identities from article 18505:

$$\begin{aligned}\frac{\delta}{\delta M_{ab}}\det M &= (M^{-1})_{ab}\det M\\ \frac{\delta}{\delta M_{ab}}(M^{-1})_{cd} &= -(M^{-1})_{ca}(M^{-1})_{bd},\end{aligned}$$

These identities hold for any invertible matrix M. Specializing these identities to the matrix whose components are the components g_{ab} of the metric tensor gives[9]

$$\frac{\delta}{\delta g_{ab}}|\det g| = |\det g|\, g^{ab} \tag{9}$$

$$\frac{\delta}{\delta g_{ab}} g^{cd}(\partial_c\phi)(\partial_d\phi') = -(\partial^a\phi)(\partial^b\phi'). \tag{10}$$

These identities can be used to show that when the action is $S = \int d^N x\,\sqrt{|\det g|}\,L$,the stress-energy tensor defined by (2) is

$$T^{ab} = -2\frac{\delta L}{\delta g_{ab}} - g^{ab}L.$$

When the lagrangian function has the specific form shown in equation (3), this becomes

$$T^{ab} = (\partial^a\phi)(\partial^b\phi) - g^{ab}\left(\frac{g^{cd}(\partial_c\phi)(\partial_d\phi)}{2} - V(\phi)\right). \tag{11}$$

When the metric is flat, this reduces to the expression shown in article 49705. This is remarkable, because the expression shown in article 49705 is based on a different definition of the stress-energy tensor! Article 32191 gives more insight into this coincidence.

[9] The symmetry of g_{ab} is imposed *after* calculating the variations. Otherwise, the diagonal components would require special treatment.

6 The covariant conservation law: example

Article 37501 explains why (and when) the general expression (2) satisfies the covariant conservation law $\nabla_a T^{ab} = 0$, where ∇ is the **Levi-Civita connection** (article 03519), which is the covariant derivative used in general relativity.[10] This section shows that the special case (11) satisfies the covariant conservation law.

The first step is to write the equation of motion (8) and the stress-energy tensor (11) in terms of the covariant derivative ∇. After this is done, the derivation of the covariant conservation law will be relatively easy, because the covariant derivative ∇ commutes with the metric g. To write (8) and (11) in terms of ∇, use these identities (article 03519):

$$\partial_a \phi = \nabla_a \phi \qquad\qquad \partial_a \left(\sqrt{|\det g|}\, V^a\right) = \sqrt{|\det g|}\, \nabla_a V^a.$$

The first identity holds for any scalar field ϕ. The second identity holds for any vector field V with components V^a, such as the combination $g^{ab}\partial_b\phi$ appearing in the equation of motion (8). Using these identities, the equation of motion (8) becomes

$$g^{ab}\nabla_a\nabla_b\phi + V'(\phi) = 0, \tag{12}$$

and the stress-energy tensor (11) becomes

$$T^{ab} = (\nabla^a\phi)(\nabla^b\phi) - g^{ab}\left(\frac{g^{cd}(\nabla_c\phi)(\nabla_d\phi)}{2} - V(\phi)\right). \tag{13}$$

Now the derivation of the covariant conservation law is relatively easy. The covariant derivative satisfies the product rule (Leibniz rule), so applying ∇_a to (13) gives

$$\nabla_a T^{ab} = (\nabla_a\nabla^a\phi)(\nabla^b\phi) + (\nabla^a\phi)(\nabla_a\nabla^b\phi) - (\nabla^c\phi)(\nabla^b\nabla_c\phi) + \nabla^b V(\phi).$$

The second and third terms on the right-hand side cancel each other, and the remainder is zero for any ϕ that satisfies the equation of motion (12).

[10] The symbol ∇ here should not be confused with the ordinary gradient with respect to the "space" coordinates, for which some other articles in this series use the same symbol ∇.

7 Why the stress-energy tensor is not unique

The definition (2) assumes that we have specified how the action depends on the background metric. Even if we only care about the flat-spacetime case, we still need to temporarily generalize the model to arbitrary not-quite-flat metrics. This can be done in more than one way, and the output of the definition (2) depends on which generalization we choose. This section shows that this dependence can persist even after specializing the result to flat spacetime.

Consider a model of a single scalar field whose action in flat spacetime is

$$S = \int d^N x \;\left(\frac{\eta^{ab}(\partial_a\phi)(\partial_b\phi)}{2} - V(\phi)\right), \tag{14}$$

where η is the Minkowski metric. To use the definition (2) of the stress-energy tensor, we need to temporarily generalize (14) to curved spacetime. To see that this generalization is not unique, consider the action

$$S = \int d^N x \,\sqrt{|\det g|}\,\left(\frac{g^{ab}(\partial_a\phi)(\partial_b\phi)}{2} - V(\phi) - \xi R\phi^2\right) \tag{15}$$

where R is the Ricci scalar (article 03519) and ξ is an arbitrary coefficient. The flat-spacetime limit of the action (15) is (14), regardless of the coefficient ξ, because $R = 0$ in flat spacetime. However, the stress-energy tensor defined by (2) still depends on ξ after making the metric flat, because[11]

$$\frac{\delta}{\delta g^{ab}}\int d^N x\;\sqrt{|\det g|}\,R\phi^2\bigg|_{g=\eta} = \partial_c\partial_d\big((\eta_{ab}\eta^{cd} - \delta_a^c\delta_b^d)\phi^2\big) \neq 0. \tag{16}$$

This shows that the stress tensor defined by (2) is not unique: it depends on a whole family of actions, parameterized by a variable metric, not just on a single action with a single specific metric.

[11] This result is shown (but not derived) in Pons (2009), section 5.3.1.

8 References

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9 References in this series

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