Why Does Smearing in Time Work Better Than Smearing in Space?

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Abstract In quantum field theory, each region of spacetime has an associated set of observables. Observables associated with individual points in smooth spacetime – or more generally with lower-dimensional submanifolds of smooth spacetime – are typically undefined as operators on a Hilbert space. Starting with a formulation that treats time as continuous but space as discrete, this article explains how to use smearing in time to construct observables whose resolution is much coarser than the discretization scale. The construction works for any renormalized operator. (This article reviews what that means.) This article also explains why smearing only in space and smearing in "euclidean spacetime" are not as effective. The general concepts are illustrated using operators constructed from a free scalar field.

Contents

1 Introduction

In quantum field theory, observables are represented by operators on a Hilbert space.[1,2] Models are often constructed by treating space or spacetime as discrete (because in most cases we don't know how to construct them directly in smooth spacetime), and then operators that are nominally localized on a lower-dimensional manifold of d-dimensional spacetime[3] may become *singular* in the smooth-spacetime limit – in the sense that they cease to be well-defined as ordinary operators on a Hilbert space. Even if the discretization scale is kept finite, a corresponding issue still exists: a given operator can represent a physically meaningful observable only if it is insensitive to discretization artifacts.[4]

Smearing[5] the operator over a region of spacetime much larger than the discretization scale can fix that problem if the original operator satisfies the *growth condition* that will be introduced in section 11. This article shows that if the original operator satisfies that condition, then smearing only in time is sufficient.[6] This article also demonstrates that smearing in space is typically not sufficient. Section 2 gives some intuition about both of those statements, and several examples will be given using scalar fields. Section 19 summarizes the examples.

Section 15 compares smearing in time to smearing in *euclidean spacetime*, which is not as effective. Sections 36-39 work through an example of this comparison.

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[1] Articles 03431 and 21916

[2] In practice, allowing operators to be defined only on a dense subset of the Hilbert space is acceptable and is often convenient. Such operators are called *unbounded* (article 74088). Example: the time translation operators $U(t)$ are defined on the whole Hilbert space, but the hamiltonian H in $U(t) = e^{-iHt}$ is unbounded. In this article, the phrase "(ordinary) operator on a Hilbert space" includes unbounded operators.

[3] This includes operators nominally localized at a point and operators nominally localized on a k-dimensional submanifold with $1 \leq k \leq d-1$.

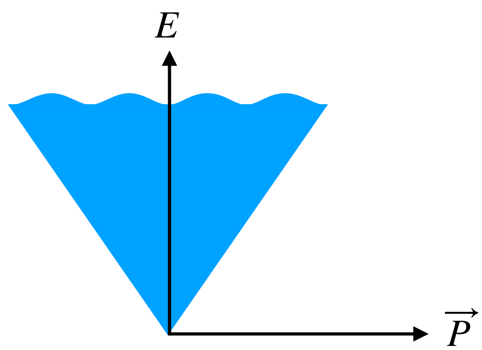
[4] This is necessary but not sufficient. Even in smooth spacetime, self-adjoint operators on the Hilbert space don't necessarily all represent observables (article 21916).

[5] Section 5 will introduce the concept of *smearing* an operator.

[6] This article only considers smearing along a prescribed time coordinate in flat spacetime, but the same idea applies to smearing along any timelike worldline in curved spacetime (Witten (2023), beginning of section 3.1).

2 Preview and intuition

Real measurements have limited resolution. In quantum theory, observables are represented by operators on a Hilbert space. The resolution in time or space of a measurement of a given observable is related to how much the operator can change the total energy or total momentum of a state on which it acts. In models with Lorentz symmetry, a general principle called the **spectrum condition** (reviewed in section 7) says that the allowed combinations of the total energy E and total momentum $\vec{P}$ of a state are restricted to the **forward light cone**, illustrated here as the blue-shaded cone:[7]

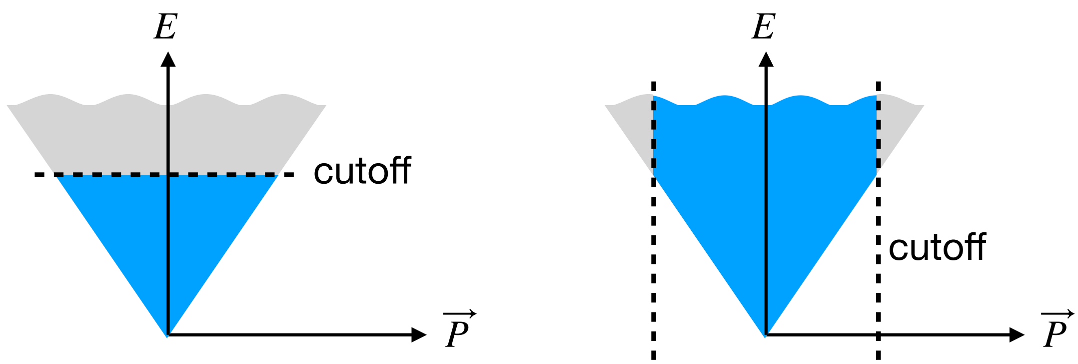


Quantum field theory is often described in smooth spacetime using operators that are nominally localized at individual points in spacetime, but those operators cannot represent physically meaningful observables because real measurements don't have infinitely fine resolution. Such "operators" are also mathematically unhealthy: they may be legitimate operators on a Hilbert space when space is discrete, but they become ill-defined in the smooth-space limit. Mathematically healthy operators with limited resolution in space and time may be constructed using **smearing** (introduced in section 5), which means replacing the original point-localized operator with an integral over displacements of that operator weighted by a **smearing function** that is concentrated in a neighborhood of nonzero size. Smearing in

[7] The wavy upper edge of the blue-shaded cone is meant to indicate that the cone extends upward to arbitrarily large energies.

time limits how much the operator can change the total energy of a state on which it acts, and smearing in space limits how much the operator can change the total momentum of a state on which it acts.

This article's main message is that if smearing in spacetime makes an operator healthy, then smearing only in time also makes it healthy, but smearing only in space typically doesn't. To understand intuitively why smearing only in time is more effective than smearing only in space, consider the superposition of energy-momentum eigenstates produced by applying the operator to a state whose energy and momentum are at the lower tip of the cone in the preceding figure. (Such a state is called a **vacuum state**.) Smearing the operator in time effectively imposes a cutoff on the maximum energy of the states in that superposition, and smearing the operator in space correspondingly imposes a cutoff on the maximum magnitude of their momenta, as illustrated here:



In each picture, energies or momenta (repsectively) retained by the cutoff are shaded in blue, and energies or momenta excluded by the cutoff are shaded in light gray. These pictures are meant to convey two key insights. First, limiting the energy automatically limits the momentum, so smearing only in time automatically limits the operator's resolution in both time and space. Second, limiting the total momentum does not limit the energy, so smearing in space only limits the operator's resolution in space, not in time. This asymmetry between the effectiveness of smearing in time and smearing in space is a consequence of the spectrum condition. Even in models without Lorentz symmetry (nonrelativistic models), a similar condition still applies. The cone with a sharp apex at the base might be replaced by a parabolic "cone" with a smooth base, but its properties are qualitatively the same: an energy bound implies a momentum bound, but not conversely.

This asymmetry has a familiar analogy in classical mechanics. Consider a system of two or more nonrelativistic and non-interacting classical particles whose individual energies are quadratic functions of their individual momenta. In that system, constraining the total momentum to be close to zero does not constrain the total energy at all, but constraining the total energy to be close to zero automatically constrains the total momentum to be close to zero. In fact, it automatically constrains each individual particle's momentum to be close to zero.

Again, this article's main message is that if smearing in spacetime makes an operator healthy, then smearing only in time also makes it healthy, but smearing only in space typically doesn't. This article explains how to make the intuition more precise and demonstrates it with several examples. The examples show that in some cases, smearing in space is sufficient,[8] but in most cases it is not, not even in a model without any interactions. The examples all use operators whose unsmeared versions are nominally localized at a single point in spacetime, but the general results in sections 11-13 are not limited to that special case.

The results in this article about the effectiveness of smearing in time apply only to algebraic products of time-smeared operators, not to time-ordered products of time-smeared operators. The path integral gives time-ordered correlation functions. Sections 15 and 39 will elaborate on this caveat.

[8]This is consistent with the classical-particle intuition, because in a system with only one particle, a bound on the total momentum does imply a bound on the total energy.

3 Outline

- Section 5 defines *smearing*.
- Section 6 establishes conditions that smearing functions should satisfy.
- Sections 7-8 review the spectrum condition.
- Section 9 introduces the no-artifacts criterion. The goal is to construct operators that satisfy this criterion.
- Sections 10-12 introduce a *growth condition* that unsmeared operators are assumed to satisfy. It amounts to assuming that smearing in spacetime would make the operator satisfy the no-artifacts criterion.
- Sections 13 shows that if an operator satisfies the growth condition, then smearing only in time should be enough to make it satisfy the no-artifacts criterion.
- Section 14 reviews related results in smooth spacetime.
- Section 15 addresses smearing in "euclidean spacetime."
- Section 16 explains what *renormalization* means for a composite operator. Renormalized composite operators provide a supply of instructive examples.
- Sections 17-18 introduce the average energy test and the norm test that will be used in the examples.
- Section 19 summarizes the examples.
- Sections 20-24 review background material about scalar fields that will be used in the examples.
- Section 25 derives an identity that will be used in some of the examples.
- Sections 26-44 work through several examples.

4 Notation and conventions

- D is number of dimensions of space, and $d \equiv D + 1$ is number of dimensions of spacetime.
- A *state* is an element of the Hilbert space (also called a state-vector).
- $|0\rangle$ is the vacuum state, normalized so $\langle 0|0\rangle = 1$.
- This article uses the hamiltonian formulation. Time is continuous. Space is a finite lattice with lattice spacing ϵ.
- The units are such that Planck's constant $\hbar$ and the speed of light are both 1, so energy has the same units as $1/\epsilon$.
- The phrase *physical units* means units that remain meaningful after $\epsilon \to 0$.
- Time translation symmetry is assumed, so a time-independent energy operator (hamiltonian) is available.
- x and y are points in spacetime, and $\mathbf{x}$ and $\mathbf{y}$ are points in space.
- $\delta(s)$ is the distribution defined by $\int ds\ \delta(s) f(s) = f(0)$.
- $U(t)$ and $U(\mathbf{x})$ denote the **translation operators**, the unitary operators that shift a state in time or space by the specified amount t or $\mathbf{x}$.

When space is discrete, $\mathcal{O}$ denotes an operator defined on (at least a dense subset of) the Hilbert space. In the smooth-space limit, $\mathcal{O}$ may or may not remain defined in that sense. $\mathcal{O}(x)$ or $\mathcal{O}(\mathbf{x}, t)$ denotes an operator localized at the indicated point in spacetime. When the point $\mathbf{x}$ or the time t is fixed by the context, the abbreviations $\mathcal{O}(t)$ or $\mathcal{O}(\mathbf{x})$ may be used. The notation $\mathcal{O}(f)$ will be used for the operator produced by smearing an operator $\mathcal{O}$, where f is the smearing function (section 5). In a model with no interactions, $(\mathcal{O})_R$ denotes the normal-ordered counterpart of $\mathcal{O}$ (section 30). The subscript R means *renormalized* (section 16), which is what normal-ordering achieves in a model with no interactions.

The Fourier transform of a function f will be denoted f^{FT}. If f is a function of the D components of $\mathbf{x}$, then its Fourier transform is a function of the D components of $\mathbf{p}$, which will be called **momentum variables**. This article uses an integral-like notation for sums over lattice sites and sums over momentum variables:[9]

$$\int_{\mathbf{x}} \cdots \equiv \epsilon^D \sum_{\mathbf{x}} \cdots \qquad\qquad \int_{\mathbf{p}} \cdots \propto \sum_{\mathbf{p}} \cdots\ .$$

In the infinite-volum limit (infinite size of the spatial lattice with fixed ϵ), the sum over $\mathbf{p}$ becomes an integral over $\mathbf{p}$, which is denoted by the same symbol:

$$\int_{\mathbf{p}} \cdots = \int \frac{d^D p}{(2\pi)^D} \cdots\ .$$

This implicitly defines the normalization of the sum. Each component of $\mathbf{p}$ is integrated over the range from $-\pi/\epsilon$ to π/ϵ, which is bounded unless the smooth-space limit $\epsilon \to 0$ is taken.

Whenever the order of sums/integrals is changed, the re-ordering is understood to be justified by the fact that the range of the sum/integral is bounded and the summand/integrand is finite.

[9] Article 71852

5 Smearing

For an observable $\mathcal{O}(x)$ that is nominally localized at a single point x in spacetime, **smearing** means replacing $\mathcal{O}(x)$ with $\int_x f(x)\mathcal{O}(x)$ for some function $f(x)$ called the **smearing function**. Instead of smearing in both time and space, this article considers the effect of smearing in only one or the other:

$$\begin{array}{lll}\text{only in time:} & \mathcal{O}(\mathbf{x},t) \to \int dt\; f(t)\mathcal{O}(\mathbf{x},t) & \int dt\; f(t) = 1 \\ \text{only in space:} & \mathcal{O}(\mathbf{x},t) \to \int_{\mathbf{x}} f(\mathbf{x})\mathcal{O}(\mathbf{x},t) & \int_{\mathbf{x}} f(\mathbf{x}) = 1.\end{array} \tag{1}$$

In the case of smearing in space, the integral is really a sum, because this article treats space as discrete. In both cases, the width of the smearing function (in time or space) should be much greater than ϵ. Section 6 will establish the properties that smearing functions should have.

If $\mathcal{O}$ is an operator localized at x and $F(\cdot)$ is a nonlinear function, then clearly

$$\text{smeared}(F(\mathcal{O})) \neq F(\text{smeared}(\mathcal{O})), \tag{2}$$

so using a language that distinguishes between the two sides of (2) can sometimes be convenient:[10]

- The operators $\mathcal{O}$ and $F(\mathcal{O})$ will be called **unsmeared**.
- The operator smeared$(F(\mathcal{O}))$ will be called **(post-)smeared**.
- The operator F(smeared$(\mathcal{O})$) will be called **pre-smeared**.

In this article, the unqualified word *smeared* means *post-smeared*.

[10]The language *post-smeared* and *pre-smeared* is not standard. The unqualified word *smeared* is standard, and it usually means *post-smeared*.

6 Required properties of smearing functions

In smooth spacetime, smearing functions are typically taken to be **Schwartz functions**.[11,12] Roughly, a function f is a Schwartz function if it decreases faster than the inverse of any polynomial as its arguments become arbitrarily large and if its Fourier transform also has that property. The function

$$f(\mathbf{x},t) \propto \exp\left(-\frac{|\mathbf{x}|^2}{\sigma_{\text{space}}^2}\right)\exp\left(-\frac{t^2}{\sigma_{\text{time}}^2}\right) \tag{3}$$

is the prototypical example of a Schwartz function.[13] Its support is unbounded, but it is concentrated in a region near $|\mathbf{x}| = 0$ with characteristic width $\sim \sigma_{\text{space}}$ and near the time $t = 0$ with characteristic width $\sim \sigma_{\text{time}}$.

Instead of smearing in both space and time, this article compares smearing only in time to smearing only in space. Examples:

$$\begin{aligned} &f(t) \propto \exp\big(-t^2/\sigma^2\big) && \text{(for smearing only in time)}\\ &f(\mathbf{x}) \propto \exp\big(-|\mathbf{x}|^2/\sigma^2\big) && \text{(for smearing only in space).} \end{aligned}$$

Equations (1) show how these functions are used.

Space will be treated as a (very fine) lattice with a (very large) finite number of points. In this context, the "arbitrarily large" regime doesn't exist: the spatial coordinates can't be arbitrarily large because the lattice is finite, and the momentum variables (the domain of the Fourier transform f^{FT} of f) can't be arbitrarily large because the lattice is discrete. In this context, a spatial smearing function should be a Schwartz function whose characteristic width σ is much less than the overall width of the lattice (so the width of the lattice is practically infinite compared to σ) but much greater than the lattice spacing ϵ (so the width $\sim 1/\epsilon$ of the momentum domain is practically infinite compared to $1/\sigma$).

[11] Haag (1996), text after equation (II.1.6)

[12] Schwartz functions are a particular class of **test functions** used in the theory of distributions (Strichartz (1994), section 3.2).

[13] Another example: every smooth (infinitely differentiable) function with compact support is a Schwartz function (Salo (2013), example 3.1.1).

7 The spectrum condition and Lorentz symmetry

This article focuses on models that have time translation symmetry. In a model with time translation symmetry, the hamiltonian H – the operator that generates translations in time – is independent of time. It represents the observable that we call the system's total energy.[14] In that context, one of the general postulates of quantum field theory requires the quantity

$$\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} \tag{4}$$

to have a finite lower bound among all states $|\psi\rangle$. This is the **spectrum condition**. For simplicity, this article assumes the existence and uniqueness of a lowest-energy state. This state will be denoted $|0\rangle$ and called the **vacuum state**.[15] The constant term in H will always be chosen so that the lower bound is zero:[16]

$$H|0\rangle = 0. \tag{5}$$

This choice is convenient in Lorentz-symmetric models, because then the spectrum condition can be expressed in a Lorentz-symmetric way as described in section 2. Section 8 will explain what happens to that view of the spectrum condition when space is discrete, which ruins exact Lorentz symmetry.

[14] By definition, the system's total energy is the conserved quantity associated with time translation symmetry. Article 12342 introduces this perspective in the context of classical mechanics.

[15] When time translation symmetry is absent, such a naturally-preferred vacuum state is also absent because the the model doesn't have a naturally-distinguished energy operator. Witten (2022) explains some implications of this in the context of cosmology.

[16] This article only considers models without gravity, so the constant term in H can be chosen arbitrarily.

8 The spectrum condition in discrete space

In this article, the important implication of the Lorentz-symmetric version of the spectrum condition is that imposing a limit on an observable's time resolution automatically limits its spatial resolution, too. This article focuses on models that would have Lorentz symmetry when discretization artifacts are neglected, so the magnitude of a state's total momentum should be bounded by its total energy after the smooth-space limit is taken. That bound does not hold before the smooth-space limit is taken, but the premise that Lorentz symmetry is a good approximation near the smooth-space limit implies that such a bound should hold to a good approximation at least for energies much less than $1/\epsilon$. (Section 22 will check this in the case of a free scalar field.) That's enough to ensure that making an observable's time resolution much coarser than ϵ is enough to make its spatial resolution much coarser than ϵ, too.

9 The no-artifacts criterion

In quantum field theory, one way to construct many models is to treat space (or spacetime) as discrete instead of smooth. If space is treated as discrete, then a scalar quantum field can be described using operators localized at individual points.[17] These operators are already well-defined as ordinary (possibly unbounded) operators on a Hilbert space without the help of smearing.

The discrete structure is artificial, though. If the distance between discrete points is $\sim \epsilon$, then the only states that can be physically meaningful are those whose total energy and total momentum (all components) are both negligible compared to $1/\epsilon$. When an operator that is meant to represent a physically meaningful observable is applied to such a state, the resulting state should still have that property.[18] Altogether, to avoid discretization artifacts, we need to do at least three things:

- Tune the parameters in the hamiltonian to make the correlation length much greater than the discretization length scale ϵ.
- Consider only states whose total energy and total momentum are $\ll 1/\epsilon$.
- Consider only observables that preserve that restriction on the states. This restriction on the observables will be called the **no-artifacts criterion**.

Smearing in spacetime is used to produce operators that satisfy the no-artifacts criterion.[19] Section 13 will show that if smearing in spacetime is sufficient, then smearing only in time is already sufficient.

[17] Article 52890

[18] This is another way to say that physically meaningful observables should not be able to resolve artificial details of the way space(time) is discretized (section 2).

[19] If the goal were to take a strict smooth-space limit, then that physical concern would also be a mathematical concern: keeping the operators well-defined as operators on a Hilbert space would require smearing them over a region of finite size in physical units. Many useful models (like quantum electrodynamics) probably don't have a nontrivial strict smooth-spacetime limit, though. The perspective used in this article applies to those models, too.

10 Precedent for the growth condition

Section 13 will derive a result about the sufficiency of smearing in time. That result relies on a *growth condition* that will be described in section 11. This section reviews a corresponding condition that is standard in the context of smooth spacetime.

One of the original axiom systems for quantum field theory in smooth spacetime uses Wightman functions. A **Wightman function** is a vacuum expectation value of a product[20] of "operators" localized at non-coincident points in spacetime, with the understanding that those point-localized entities become legitimate operators on a Hilbert space only when they are smeared in spacetime.[21] When expressed precisely, this premise about the effectiveness of smearing in spacetime implies that an unsmeared Wightman function is a **tempered distribution**,[22,23] which in turn implies[24] a bound on how rapidly the function's Fourier transform can grow as a function of energy and momentum – the Fourier duals of the time and space coordinates. Thanks to this bound, smearing in spacetime has the effect of imposing a (smooth version of a) cutoff in energy and momentum, like section 2 described intuitively. If the growth were too rapid, then smearing in spacetime would not be effective.[25]

In this article, space is treated as discrete instead of smooth.[26] In this context,

[20]Wightman functions use the ordinary algebraic product, not the time-ordered product (section 14).

[21]Streater and Wightman (1980), chapter 3; Haag (1996), sections II.1.2 and II.2.1

[22]Streater and Wightman (1980), text after equation (3-20)

[23]Strichartz (1994) gives a proper introduction to the concept of a *tempered distribution*. Article 58590 gives a quick review.

[24]Article 58590

[25]Wightman functions must be tempered distributions if the set of allowed smearing functions "consists of all complex-valued, infinitely differentiable functions f, which, together with their derivatives, approach zero at infinity faster than any power of the euclidean distance [defined to be the sum of the squares of the coordinates]" (Streater and Wightman (1980), text before equation (2-4)). Jaffe (1967) considers a more restricted set of smearing functions. This allows the Fourier transforms of unsmeared correlation functions to grow more rapidly as a function of energy and momentum. This is reviewed in Buoninfante *et al* (2024), section 2.1, text around equations (5)-(7). Jaffe (1966) lists some examples of allowed growth rates. Still, if the growth is *too* rapid, then smearing would not be effective.

[26]The name *Wightman function* is reserved for correlation functions in smooth spacetime. This article uses more generic name *correlation function* for a vacuum expectation value of a product of operators in not-necessarily-smooth spacetime. The product is still the algebraic product, not the time-ordered product (footnote 20).

smearing is used to enforce the no-artifacts criterion – to construct such observables by limiting their resolution in space and time, of equivalently by limiting how much they can change the energy and momentum of any states on which they act.[27] Just like in the smooth-spacetime context, this only works if the unsmeared operator $\mathcal{O}$ already satisfies a bound on how quickly certain quantities constructed from $\mathcal{O}$ can grow as a function of energy and momentum. Section 11 will introduce such a *growth condition* tailored for the context of discrete space. Section 12 will justify this condition by showing that if $\mathcal{O}$ satisfies it, then smearing $\mathcal{O}$ in spacetime (with a suitable smearing function) enforces the no-artifacts criterion. This is analogous to the situation in smooth spacetime reviewed in the previous paragraph.

[27] Section 9

11 The growth condition

Let $\mathcal{O}$ be an operator that may be localized at a point in spacetime.[28] This section introduces a *growth condition*, a condition on the rate at which certain quantities involving $\mathcal{O}$ can grow as a function of energy and momentum. Section 12 will justify the condition by showing that if $\mathcal{O}$ satisfies it, then smearing $\mathcal{O}$ in spacetime with an appropriate smearing function enforces the no-artifacts criterion.[29] Section 13 will use the same growth condition to show that smearing $\mathcal{O}$ only in time is already sufficient.

Let $|j\rangle$ and $|k\rangle$ be two eigenstates of the total energy and momentum operators. The total energy and momentum of the state $|j\rangle$ will be denoted E_j and $\mathbf{p}_j$, respectively.[30] Then any operator $\mathcal{O}$ may be expressed as

$$\mathcal{O} = \sum_{j,k} |j\rangle\, \mathcal{O}_{jk}\, \langle k|, \tag{6}$$

where the sum is over a complete basis of energy-momentum eigenstates and

$$\mathcal{O}_{jk} \equiv \langle j|\mathcal{O}|k\rangle. \tag{7}$$

The quantities $\mathcal{O}_{jk}$ will be called the **matrix elements** of $\mathcal{O}$ in the energy-momentum basis. In this article, saying that the operator $\mathcal{O}$ satisfies the **growth condition** means that for any given eigenstate $|k\rangle$ whose energy and momentum are much less than $1/\epsilon$, the magnitude of $\mathcal{O}_{jk}$ does not grow any faster than a polynomial in E_j and the components of $\mathbf{p}_j$. In symbols, the growth condition is

$$|\mathcal{O}_{jk}| < c + E_j^N + |\mathbf{p}_j|^N \qquad\qquad \text{for all } j \tag{8}$$

for some sufficiently large constant c and sufficiently large integer N, both of which may depend on the state $|k\rangle$.

[28] The analysis in sections 11-13 holds whether or not $\mathcal{O}$ is localized at a point.

[29] Section 9

[30] For each j, $\mathbf{p}_j$ has D components, where D is the number of dimensions of space.

12 Justification for the growth condition

This section shows that if an operator $\mathcal{O}$ satisfies the growth condition introduced in section 11, then smearing $\mathcal{O}$ in spacetime with an appropriate smearing function enforces the no-artifacts criterion.

Let $\mathcal{O}$ be an operator localized at a point in spacetime, and let $U(\mathbf{x})$ and $U(t)$ be the unitary operators that translate a state by the specified amounts in space and time, respectively. The position- and time-dependent version of $\mathcal{O}$ is

$$\mathcal{O}(\mathbf{x},t) \equiv U(-t)U(-\mathbf{x})\mathcal{O}U(\mathbf{x})U(t).$$

The operator obtained by smearing $\mathcal{O}$ in both space and time is

$$\mathcal{O}(f) \equiv \int dt\, \int_{\mathbf{x}} f(\mathbf{x},t)\mathcal{O}(\mathbf{x},t),$$

where f is the smearing function. Define $\mathcal{O}_{jk}$ and $|j\rangle$ and $|k\rangle$ as in section 11. Use $U(\mathbf{x},t)|k\rangle = e^{-iE_k t+i\mathbf{p}_k\cdot\mathbf{x}}|k\rangle$ to get

$$\begin{aligned}\langle j|\mathcal{O}(f)|k\rangle &= \int dt\, \int_{\mathbf{x}} f(\mathbf{x},t)\, e^{i(E_j-E_k)t+i(\mathbf{p}_j-\mathbf{p}_k)\cdot\mathbf{x}}\, \mathcal{O}_{jk} \\ &= f^{\text{FT}}(E_j-E_k,\, \mathbf{p}_j-\mathbf{p}_k)\, \mathcal{O}_{jk}\end{aligned} \tag{9}$$

where f^{FT} is the Fourier transform of the smearing function f. Example: if

$$f(\mathbf{x},t) \propto e^{-(t^2+\mathbf{x}^2)/2\sigma^2},$$

then

$$\langle j|\mathcal{O}(f)|k\rangle \propto e^{-((E_j-E_k)^2+(\mathbf{p}_j-\mathbf{p}_k)^2)\sigma^2/2}\, \mathcal{O}_{jk}.$$

Equations (6) and (9) show that the operator $\mathcal{O}(f)$ may be written

$$\mathcal{O}(f) = \sum_{j,k} f^{\text{FT}}(E_j-E_k,\, \mathbf{p}_j-\mathbf{p}_k)\, \mathcal{O}_{jk}|j\rangle\,\langle k|. \tag{10}$$

The growth condition (8) and the properties of the smearing function established in section 6 imply that applying the operator (10) to a state whose total energy and total momentum are much smaller than $1/\epsilon$ gives another state with that property. In other words, $\mathcal{O}(f)$ satisfies the no-artifacts criterion.

13 The sufficiency of smearing in time

Section 12 showed that if an operator $\mathcal{O}$ satisfies the growth condition established in section 11, then the operator obtained by smearing $\mathcal{O}$ in spacetime satisfies the no-artifacts criterion. This section shows that the operator obtained by smearing $\mathcal{O}$ only in time already satisfies the no-artifacts criterion. In other words, smearing $\mathcal{O}$ only in time is enough to produce a physically meaningful observable.

The time-smeared version of $\mathcal{O}$ is

$$\mathcal{O}(f) \equiv \int dt\ f(t)U(-t)\mathcal{O}U(t). \tag{11}$$

Following the same steps as in section 12 gives

$$\mathcal{O}(f) = \sum_{j,k} f^{\text{FT}}(E_j - E_k)\,\mathcal{O}_{jk}|j\rangle\,\langle k|. \tag{12}$$

Example: using the smearing function $f(t) \propto e^{-t^2/2\sigma^2}$ gives

$$\mathcal{O}(f) \propto \sum_{j,k} e^{-(E_k-E_j)^2\sigma^2/2}\mathcal{O}_{jk}|j\rangle\,\langle k|.$$

Thanks to the growth condition (8), if the width of the smearing function is $\gg \epsilon$, then applying $\mathcal{O}(f)$ to a state $|s\rangle$ with total energy $\ll 1/\epsilon$ gives another state $\mathcal{O}(f)|s\rangle$ with that property.

The goal is to show that if the total momentum of $|s\rangle$ is also $\ll 1/\epsilon$, then $\mathcal{O}(f)|s\rangle$ has that property, too. As anticipated by the intuition in section 2, this is a consequence of the spectrum condition. The spectrum condition says[31] that if $E \ll 1/\epsilon$, then the total momentum $\mathbf{p}$ and total energy E of a state satisfy $|\mathbf{p}| \lesssim E$, so if total energy of the state $\mathcal{O}(f)|s\rangle$ is much less than $1/\epsilon$, then its total momentum must be, too. This is why smearing only in time is sufficient.

[31] Section 8

14 Related results in smooth spacetime

The result in section 13 applies to models defined in discrete space that may or may not have a nontrivial strict smooth-space limit. This section reviews corresponding results in the context of smooth space. In this context, the question is whether point-localized entities like those appearing in Wightman functions[32] can be turned into legitimate operators on a Hilbert space.

One theorem about the sufficiency of smearing in time was published in Borchers (1964). That result refers to the point-localized entities in Wightman functions, which are assumed to be tempered distributions.

A similar result is reviewed in Witten (2023). The conclusion is summarized by this excerpt:[33,34]

> Though smearing in space is only effective in favorable cases, smearing in real time [along any timelike worldline] turns a "local operator" of any dimension into a true operator.

That result is derived from an assumption about the coefficients in the **operator product expansion**,[35] which in turn implies a bound on the rate of growth in the energy-momentum domain similar to the one satisfied by Wightman functions. Either conformal invariance or asymptotic freedom would be sufficient to imply the assumed property.[36]

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[32] Section 10

[33] Witten (2023), text before equation (2.3)

[34] In this excerpt, "local operator" refers to something localized at a point in spacetime. Witten (2023) emphasizes the fact that in smooth spacetime, such a thing is not really an *operator* in the sense of a conventional linear operator on a Hilbert space (the text before equation (2.1) says "a local operator is not really a Hilbert space operator"). The purpose of smearing the "operator" in time is to repair that flaw.

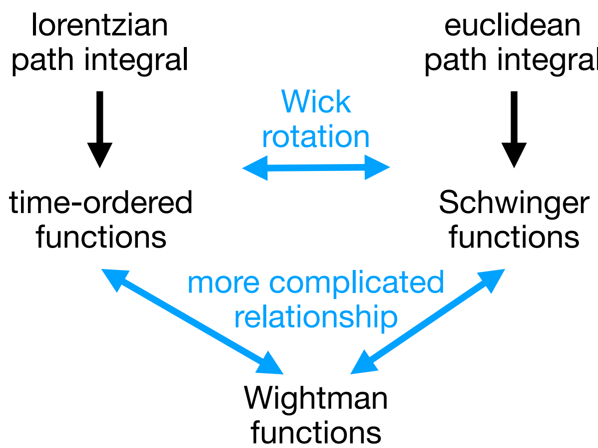
[35] Wilson and Zimmermann (1972) gives a relatively concise introduction to the ideas behind the operator product expansion.

[36] Witten (2023), text after equation (2.6)

15 Smearing in euclidean spacetime

Witten (2023) points out that smearing in d-dimensional *euclidean* spacetime is only as effective as smearing in d-dimensional space would be,[37] not as effective as smearing in "real time." This section explains what that means and offers some intuition about why it is true.

Consider the three different types of correlation function shown here:



The path integral formulation naturally produces either time-ordered correlation functions[38] or **Schwinger functions**. These are related to each other by Wick rotation, which converts the signature of the spacetime metric from lorentzian to euclidean or conversely. These two types of correlation function are both invariant under permutations of the operators in the "product," up to an overall sign depending on fermion numbers. Wightman functions (featured in sections 10 and 14) don't have that property – except when the operators in the product commute or anticommute with each other – because Wightman functions use the ordinary algebraic product, not the time-ordered "product."[39] The term "real time" used in

[37] Witten (2023), text between equations (2.2) and (2.3)

[38] Article 63548

[39] Sections 1.3.1-1.3.2 in Montvay and Münster (1997) review the more-complicated analytic relationship between Wightman functions and the other two types.

Witten (2023) refers to Wightman functions and to the operator product expansion, both of which use the algebraic (not time-ordered) product.

The difference between algebraic products and time-ordered products is clearly important when considering operators that are smeared in time if the supports of their smearing functions overlap. The results in this article about the effectiveness of smearing in time apply when considering algebraic products.[40] In particular, the average energy test and the norm test use algebraic products. Sections 36-39 will show an example of an operator that passes those tests because it's smeared in time but fails modified versions of those tests that use time-ordered products instead of algebraic products. The factor that determines the effectiveness of smearing is whether algebraic products are used versus time-ordered products, not whether the signature is lorentzian versus euclidean, because Wightman functions and time-ordered functions both use lorentzian signature. Saying that smearing in euclidean time is not effective is really just an indirect (via Wick rotation) way of saying that time-ordering disrupts the benefits of smearing in time, which is relatively unsurprising.

In this article's perspective, the purpose of smearing in time is to produce an operator that can represent a physically meaningful observable – an observable that, if measured, would not resolve any artificial details of the space-discretization scheme that was used only as a compromise to define the model. If we emulate the effect of a measurement by projecting the state onto one of the observable's eigenspaces, then the algebraic product is clearly the relevant one: the observable must be smeared in time before the projection is applied. That simplistic way of describing measurements is the context for this article. Ideally, though, we would treat the measurement as a physical process described within the theory itself.[41] In that case, instead of choosing an observable, we would only choose an initial state, and time-evolution would determine which (if any) observable ends up being measured. Using an initial state that is smeared in time (equivalently, one that involves only energies much less than $1/\epsilon$) would avoid discretization artifacts.

[40] If the smearing functions have compact supports that don't overlap, then the time-ordered product is an ordinary algebraic product, so the results do apply in that case.

[41] Article 03431

16 Renormalized composite operators

The intuition in section 2 suggests that smearing in space won't always work even when smearing in time does work. Sections 26-44 will demonstrate this through several examples. Most of the examples use **composite operators**, operators that involve products of the model's basic field operators.[42] Most composite operators have an ailment that smearing does not cure:[43] correlation functions of point-localized composite operators typically become undefined (or exhibit other forms of bad behavior)[44] in the smooth-space limit, even when the points at which the operators are localized remain distinct from each other in that limit. Modifying the operator to cure this ailment without changing its point-localized status is called **renormalizing** the composite operator.[45,46] To help prevent mistaking one ailment for another, the examples in this article that use composite operators will use *renormalized* composite operators.

In models with interactions, renormalized composite operators are usually only constructed order-by-order in a small-interaction expansion.[47] To avoid dealing with small-interaction expansions,[48] most of the examples in this article use a model with no interactions at all. In such a model with no interactions, renormalized composite operators may be constructed exactly using **normal ordering**, which is reviewed in section 30.

[42]Collins (1984), chapter 6

[43]Smearing can cure divergences that occur when two of those points coincide, but if a correlation function is badly behaved even when its points don't coincide, then smearing isn't enough.

[44]Section 41

[45]Renormalizing the model itself means making the parameters in hamiltonian (or action) functions of ϵ so that the model's low-energy predictions are practically independent of ϵ. Renormalizing a composite operator does not involve adjusting the parameters in the hamiltonian (or action).

[46]Renormalized composite operators are the operators featured in the *operator product expansion* in smooth space (section 14).

[47]Witten (1996), lecture 3; Collins (1984), chapter 6; McGreevy (2017), end of section 9.3.2

[48]I don't know how to extract definitive conclusions about things like the average energy test or norm test from the first few terms in a small-interaction expansion.

17 Testing an operator's health

This section describes two tests, the average energy test and the norm test,[49] that sections 26-44 will use to check the health of the operators in several examples. These tests are not perfect. In particular, section 18 will emphasize that the norm test can easily be deceived. The virtue of these tests is they are relatively easy to check.

If an observable $\mathcal{O}$ satisfies the no-artifacts criterion, then

$$\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} \ll \frac{1}{\epsilon} \qquad\qquad \text{with } |\psi\rangle = \mathcal{O}|0\rangle. \tag{13}$$

This inequality will be called the **average energy test**. If an operator $\mathcal{O}$ fails this test, then it violates the no-artifacts criterion.

The fact that an operator smeared in time passes the average energy test is not surprising, so the average energy test will be supplemented by another test that is sensitive to spatial resolution, too.[50] To motivate the test, let $U(\mathbf{x})$ denote the unitary operator that shifts a state in space by the specified amount $\mathbf{x}$, and use the abbreviation $|\psi\rangle \equiv \mathcal{O}|0\rangle$. The smooth-spacetime limit of the correlation function $\langle\psi|U(\mathbf{x})|\psi\rangle$ may diverge as $\mathbf{x} \to \mathbf{0}$ even if it is finite for all nonzero values of $\mathbf{x}$. This is the expected behavior when $\mathcal{O}$ is localized at a point. On the other hand, if the smooth-space limit of $\mathcal{O}$ is a well-defined operator, then the smooth-spacetime limit of the correlation function should be finite for all $\mathbf{x}$, including $\mathbf{x} = \mathbf{0}$. That motivates the **norm test**, which tries to diagnose the operator's health based on whether the quantity $\langle\psi|\psi\rangle$ (the correlation function at $\mathbf{x} = 0$) remains finite in the smooth-spacetime limit.

[49]These names are not standard, but the norm test is commonly used (example: Witten (2023), text around equation (2.1)).

[50]Section 18 will explain the sense in which the norm test is sensitive to resolution in space.

18 How to deceive the norm test

An operator that passes the norm test is not necessarily healthy according to the criteria in sections 9 and 16. We can make any operator $\mathcal{O}$ pass the test by replacing it with $\mathcal{O}/\sqrt{\langle\psi|\psi\rangle}$ before taking the smooth-spacetime limit.[51,52] This is deceptive because it doesn't improve the operator's health at all: the no-artifacts criterion is independent of the operator's overall normalization. Section 40 will show that this type of deception can occur unintentionally.

In the case of a point-localized operator, that type of deception can be caught by considering the correlation function $\langle\psi|U(\mathbf{x})|\psi\rangle$: if the smooth-spacetime limit of the correlation function drops suddenly to zero as soon as $\mathbf{x} \neq \mathbf{0}$,[53] then clearly the operator is not healthy, because an operator that satisfies the no-artifacts criterion cannot have infinitely fine resolution in space.[54]

A more devious version of that deception starts by smearing the operator in space, say with a smearing function of the form $\propto e^{-\mathbf{x}^2/2\sigma^2}$. Then the correlation function $\langle 0|\mathcal{O}U(\mathbf{x})\mathcal{O}|0\rangle$ has nonzero width ($\geq \sigma$) even after normalizing it to have a finite smooth-space limit at $\mathbf{x} = 0$, but the operator itself may still be unhealthy: it may still violate the no-artifacts criterion. Section 33 will demonstrate this.

[51] If the operator is meant to represent an observable, then merely multiplying it by an overall factor has no effect on the corresponding observable: the way we label the possible outcomes of a measurement has no effect on the set of possible outcomes (article 03431). Still, normalization factors that diverge in the smooth-spacetime limit cause mathematical problems in that limit.

[52] The opposite type of deception is also possible: a healthy operator can be made to fail the norm test by applying a divergent normalization factor, even though such a factor has no effect on the no-artifacts criterion.

[53] The smooth-spacetime limit of a correlation function of point-localized operators is typically finite and nonzero when the points are distinct and diverges when two or more points coincide. In discrete spacetime, instead of diverging, its peak value is very large but finite. If the function is normalized to make its peak value equal to 1 before taking the smooth-spacetime limit, then (if that normalization condition is maintained) the peak will have zero width after taking the smooth-spacetime limit.

[54] Section 41 will mention another reason why such a correlation function is not acceptable: it would imply that the Hilbert space is not separable.

19 Summary of the examples

The rest of this article focuses on examples. This table summarizes the examples:

section	model	operator	**smearing type**	energy test	norm test
26	free	ϕ	none	fail	fail
27 & 28	free	ϕ	space	pass	pass
29	int	ϕ	space	pass	pass
30	free	ϕ^n	none	correlations bad	
30	free	$(\phi^n)_R$	none	correlations good	
31 & 32	free	$(\phi^2)_R$	space		fail
33	free	$(\phi^2)_R$	space	fail	
34	free	$(\phi^n)_R$	space	fail	fail
36 & 37	free	$(\phi^n)_R$	time		pass
38	free	$(\phi^n)_R$	time	pass	
40	free	$e^{ic\phi}$	none	fail	pass
41	free	$e^{ic\phi}$	none	correlations bad	
42	free	$(e^{ic\phi})_R$	none	correlations good	
42	free	$(e^{ic\phi})_R$	none		fail
43	free	$(e^{ic\phi})_R$	space		fail
44	free	$(e^{ic\phi})_R$	time	pass	pass

Notes:

- In the second column, the abbreviation "int" means "interacting."
- $(\mathcal{O})_R$ is a renormalized version of $\mathcal{O}$. Renormalization is used to construct point-localized operators that satisfy the correlation condition described in section 16.
- An empty entry in one of the last two columns means the specified section does not check that test. Sections with "correlations good/bad" in those columns check the correlation condition described in section 16.

20 Review of the quantum scalar field

Consider a quantum scalar field, using the formulation and notation from article 52890. In that formulation, time is continuous and space is discrete. The model's basic observables are the scalar field operators $\phi(\mathbf{x},t)$ and their canonical conjugates $\dot\phi(\mathbf{x},t)$. By definition, $\phi(\mathbf{x},t)$ is localized at $\mathbf{x}$ at time t. In discrete space, these satisfy the equal-time commutation relation

$$\big[\phi(\mathbf{x},t),\dot\phi(\mathbf{y},t)\big] = \frac{i}{\epsilon^D}\times\begin{cases}1 & \text{if } \mathbf{x}=\mathbf{y},\\ 0 & \text{otherwise.}\end{cases} \tag{14}$$

Choose a time t and let $[\phi]$ denote the set of operators $\phi(\mathbf{x},t)$ at that time. Let $V[\phi]$ be a function that depends only on the field ϕ at that time, not on $\dot\phi$. The hamiltonian (energy operator) has the form[55]

$$H = \epsilon^D\sum_{\mathbf{x}}\dot\phi^2(\mathbf{x},t)/2 + V[\phi]. \tag{15}$$

The sum is over all points $\mathbf{x}$ in space at a fixed time t. The time translation operators are $U(\Delta t)=e^{-iH\,\Delta t}$, which implies $\dot\phi(\mathbf{x},t)=d\phi(\mathbf{x},t)/dt$. The hamiltonian H is invariant under translations in time because it commutes with itself.

The function V is chosen so that the model has a nonzero mass gap (nonzero single-particle mass) and so that $\phi\to-\phi$ is a symmetry of the model that is not spontaneously broken.[56,57] The constant term in V is chosen to enforce (5). Let $|0\rangle$ denote the vacuum (lowest-energy) state. The symmetry $\phi\to-\phi$ implies

$$\langle 0|\dot\phi(x)|0\rangle = 0. \tag{16}$$

[55] Article 52890

[56] Article 07246

[57] If the symmetry were spontaneously broken, then many states would have the same lowest energy in the smooth-spacetime limit, and the ones that satisfy the **cluster property** wouldn't satisfy $\langle 0|\phi(x)|0\rangle = 0$ (article 21916).

21 Review of the free scalar field

A scalar field is called **free** or **non-interacting** if the V in equation (15) is a quadratic polynomial in the operators ϕ (not $\dot{\phi}$) at time t, so

$$V[\lambda\phi] = \lambda^2 V[\phi] + \text{constant}$$

for all positive real numbers λ, where the constant term is proportional to the identity operator. If V is chosen so the model becomes Lorentz symmetric in the smooth-space limit, then the field operators may be written

$$\phi(x) = \phi^+(x) + \phi^-(x) \tag{17}$$

with[58]

$$\begin{aligned} \phi^+(\mathbf{x},t) &= \int_{\mathbf{p}} \frac{e^{i\omega(\mathbf{p})t - i\mathbf{p}\cdot\mathbf{x}}\, a^\dagger(\mathbf{p})}{\sqrt{2\omega(\mathbf{p})}} \\ \phi^-(\mathbf{x},t) &= \int_{\mathbf{p}} \frac{e^{-i\omega(\mathbf{p})t + i\mathbf{p}\cdot\mathbf{x}}\, a(\mathbf{p})}{\sqrt{2\omega(\mathbf{p})}}, \end{aligned} \tag{18}$$

where the operators $a(\mathbf{p})$ satisfy

$$\big[a(\mathbf{p}), \phi(\mathbf{x},t)\big] = \frac{e^{i\omega(\mathbf{p})t - i\mathbf{p}\cdot\mathbf{x}}}{\sqrt{2\omega(\mathbf{p})}} \qquad\qquad a(\mathbf{p})|0\rangle = 0. \tag{19}$$

The $\epsilon \to 0$ limit of the function $\omega(\mathbf{p})$ is $\sqrt{m^2 + \mathbf{p}^2}$, where m is the single-particle mass. The integrals in equations (18) are really sums with a finite number of terms. They become integrals in the infinite-volume limit, as the notation suggests, but the integration range for each component of $\mathbf{p}$ is still bounded.[59]

We can use equations (17)-(19) to get useful expressions for any correlation function of the field operators. Section 23 reviews an example.

[58] Article 00980

[59] Section 4

22 The spectrum condition in discrete space

This article focuses on models that would have Lorentz symmetry when discretization artifacts are neglected. If Lorentz symmetry were exact, then the magnitude of a state's total momentum would be bounded by its total energy. That bound does not hold before the smooth-space limit is taken, but the premise that Lorentz symmetry is a good approximation near the smooth-space limit implies that such a bound should hold to a good approximation at least for energies much less than $1/\epsilon$.[60] This section shows that it does in the case of free scalar field, where the calculations can be done exactly.

Each application of the operator $a^\dagger(\mathbf{p})$ defined in section 21 increases the total energy of the state by $\omega(\mathbf{p})$ and adds the vector $\mathbf{p}$ to its total momentum. The $\epsilon \to 0$ limit of the function $\omega(\mathbf{p})$ is $\sqrt{m^2+\mathbf{p}^2}$, so in that limit, the magnitude of the total momentum cannot exceed the magnitude of the total energy. The question is what happens to that inequality when ϵ is nonzero.

When ϵ is nonzero, the function $\omega(\mathbf{p})$ is[61]

$$\omega(\mathbf{p}) = \sqrt{m^2 + \sum_{k=1}^{D}\left(\frac{2\sin(p_k\epsilon/2)}{\epsilon}\right)^2}$$

where p_k is the kth component of $\mathbf{p}$. The magnitude of p_k is restricted to be $\leq \pi/\epsilon$. If one component of $\mathbf{p}$ is π/ϵ and the others are zero, then $\omega(\mathbf{p}) = \sqrt{m^2+4/\epsilon^2} \approx 2/\epsilon$ for $m \ll 1/\epsilon$. This shows that the magnitude of the total momentum can exceed the magnitude of the total energy, but only by a factor of order 1.

A physically meaningful state should only involve energies $\ll 1/\epsilon$, so a more relevant question is what happens to the inequality when the total energy is $\ll 1/\epsilon$. Suppose the total energy of a state by applying n factors of $a^\dagger(\mathbf{p})$ to the vacuum state is less than a cutoff $\Lambda \ll 1/\epsilon$. Then the total momentum $n\mathbf{p}$ is subject to the

[60] Section 8

[61] Article 00980

inequality

$$n\omega(\mathbf{p}) < \Lambda \ll \frac{1}{\epsilon}.$$

Explicitly,

$$n\sqrt{m^2 + \sum_{k=1}^{D}\left(\frac{2\sin(p_k\epsilon/2)}{\epsilon}\right)^2} < \Lambda \ll \frac{1}{\epsilon}.$$

This justifies using the small-θ approximation $\sin\theta \approx \theta$ to get

$$n\sqrt{m^2 + \mathbf{p}^2} < \Lambda.$$

This says that if the total energy is $\ll 1/\epsilon$, then the magnitude of the total momentum does not exceed the total energy, just like if Lorentz symmetry were exact.

23 Free scalar field: equal-time 2-point function

Let ϕ be a free scalar field, and consider the equal-time 2-point correlation function

$$g(\mathbf{x}-\mathbf{y}) \equiv \langle 0|\phi(\mathbf{x},t)\phi(\mathbf{y},t)|0\rangle \tag{20}$$

in the infinite-volume limit. The commutation relation $[\phi(\mathbf{x},t),\phi(\mathbf{y},t)] = 0$ implies that the function (20) is real-valued and symmetric. We can use equations (17)-(19) to derive

$$g(\mathbf{x}) = \int_{\mathbf{p}} \frac{e^{-i\mathbf{p}\cdot\mathbf{x}}}{\omega(\mathbf{p})}. \tag{21}$$

Section 24 will deduce a useful approximation to this function.

Let $\phi^{\pm}(\mathbf{x})$ denote the energy raising/lowering parts of $\phi(\mathbf{x})$ (equation (17)). Their commutator is proportional to the identity operator I:

$$[\phi^-(\mathbf{x}),\phi^+(\mathbf{y})] = g(\mathbf{x}-\mathbf{y})I \tag{22}$$

with g defined by (20).

24 The 2-point function with coincident points

The integral (21) is well-defined because the integration range for $\mathbf{p}$ is finite. We can emulate this by replacing the original integral with

$$g(\mathbf{x}) \underset{\sim}{\propto} \int_{\mathbf{p}} \frac{e^{-i\mathbf{p}\cdot\mathbf{x}}}{\omega(\mathbf{p})}\, e^{-c\,\epsilon^2\mathbf{p}^2} \tag{23}$$

for some $c \sim 1$, where now the integration domain is unbounded. Thanks to rotation symmetry, this depends only on $|\mathbf{x}|$. For $|\mathbf{x}|$ in the range $\epsilon \ll |\mathbf{x}| \ll 1/m$, the integral is approximately independent of ϵ and m, so we can use dimensional analysis to get[62]

$$g(\mathbf{x}) \underset{\sim}{\propto} \frac{1}{|\mathbf{x}|^{D-1}} \qquad\qquad \epsilon \ll |\mathbf{x}| \ll \frac{1}{m} \text{ and } D \geq 2 \tag{24}$$

For $|\mathbf{x}| = 0$, the integral is still approximately independent of m, so dimensional analysis gives[63]

$$g(\mathbf{0}) \underset{\sim}{\propto} \frac{1}{\epsilon^{D-1}} \qquad\qquad D \geq 2. \tag{25}$$

[62]For $|\mathbf{x}| \gg 1/m$, article 22050 shows that $g(\mathbf{x})$ approaches zero like $e^{-m|\mathbf{x}|}$, ignoring factors like $1/|\mathbf{x}|^{D-1}$ that decrease less rapidly.

[63]We could have anticipated this more quickly using the definition (20) and the fact that the field $\phi(\mathbf{x})$ has the same units as $\epsilon^{(1-D)/2}$.

25 The average energy test: a useful identity

This section derives an identity that will be used in some of the examples.

Consider an operator $\mathcal{O}(x)$ localized at x in discrete spacetime, and consider its smeared version $\mathcal{O}(f)$. The smearing can be in spacetime, or only in space, or only in time. Suppose that $\mathcal{O}(f)$ is hermitian and that the smearing function f is real-valued, so $\mathcal{O}^\dagger(f) = \mathcal{O}(f)$. Use $H|0\rangle = 0$[64] and the abbreviation[65]

$$\dot{\mathcal{O}}(f) \equiv i[H, \mathcal{O}(f)] \tag{26}$$

to get this expression for the energy of the state $|\psi\rangle \equiv \mathcal{O}(f)|0\rangle$:

$$\begin{aligned}\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} &= \frac{\big\langle 0\big|[\mathcal{O}(f), H]\mathcal{O}(f)\big|0\big\rangle + \big\langle 0\big|\mathcal{O}(f)[H, \mathcal{O}(f)]\big|0\big\rangle}{2\langle\psi|\psi\rangle} \\ &= i\frac{\langle 0|\,\big[\dot{\mathcal{O}}(f),\, \mathcal{O}(f)\big]\,|0\rangle}{2\big\langle 0\big|\mathcal{O}^2(f)\big|0\big\rangle}.\end{aligned} \tag{27}$$

[64]Equation (5)

[65]H is the generator of time-evolution, and $\dot{\mathcal{O}}$ is the derivative of $e^{iHt}\mathcal{O}e^{-iHt}$ with respect to t evaluated at $t = 0$.

26 Free field without smearing

This section shows that without any smearing, the free scalar field operator $\phi(\mathbf{x},t)$ fails the norm test and the average energy test.

Equations (20) and (25) show that the unsmeared scalar field operator $\phi(\mathbf{x},t)$ fails the norm test, because (25) diverges as $\epsilon\to 0$.

For the average energy test, choose a point $\mathbf{x},t$ in spacetime and define $|\psi\rangle\equiv\phi(\mathbf{x},t)|0\rangle$. This has finite norm because space is discrete: the limit $\epsilon\to 0$ is not taken yet. Use this and the equal-time commutation relation (14) in the identity (27) to get

$$\begin{aligned}\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} &= \frac{\big\langle 0\big|[\phi(\mathbf{x},t),H]\phi(\mathbf{x},t)\big|0\big\rangle+\big\langle 0\big|\phi(\mathbf{x},t)[H,\phi(\mathbf{x},t)]\big|0\big\rangle}{2\langle\psi|\psi\rangle}\\ &= \frac{i\,\big\langle 0\big|[\dot\phi(\mathbf{x},t),\phi(\mathbf{x},t)]\big|0\big\rangle}{2\langle 0|\phi^2(\mathbf{x},t)|0\rangle}\\ &= \frac{1}{2\epsilon^D\langle 0|\phi^2(\mathbf{x},t)|0\rangle}.\end{aligned}$$

Using equation (25) to evaluate $\langle\psi|\psi\rangle$ gives

$$\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}\gtrsim\frac{1}{\epsilon}\ \text{as}\ \epsilon\to 0, \tag{28}$$

so $\phi(\mathbf{x},t)$ fails the average energy test.

27 Free field smeared in space

This section shows that the free scalar field operator $\phi(f)$ smeared only in space passes both the norm test and the average energy test.

Use equation (21) to deduce[66]

$$\langle 0|\phi^2(f)|0\rangle \underset{\sim}{\propto} \int_{\mathbf{p}} \frac{|f^{\rm FT}(\mathbf{p})|^2}{\omega(\mathbf{p})} \tag{29}$$

where $f^{\rm FT}$ is the Fourier transform of the smearing function f. Equation (1) implies that the integral

$$\int_{\mathbf{p}} |f^{\rm FT}(\mathbf{p})|^2$$

remains finite as $\epsilon \to 0$,[67,68] so the integral (29) does, too. This shows that smearing only in space suffices to make the free scalar field operator pass the norm test.

If $|\psi\rangle = \phi(f)|0\rangle$, then the quantity (27) is

$$\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = i\frac{\langle 0|[\dot\phi(f),\phi(f)]|0\rangle}{2\langle 0|\phi^2(f)|0\rangle} \underset{\sim}{\propto} \frac{\int_{\mathbf{x}} |f(\mathbf{x})|^2}{\int_{\mathbf{p}} |f^{\rm FT}(\mathbf{p})|^2/\omega(\mathbf{p})}, \tag{30}$$

which is also finite as $\epsilon \to 0$. This shows that smearing only in space suffices to make the free scalar field operator pass the average energy test. This is consistent with the intuition in section 2, because applying the free scalar field operator to the vacuum state produces a single-particle state.[69]

---

[66] Throughout these examples, all unwritten proportionality factors are independent of ϵ.

[67] Hunter (2005), theorem 11.37

[68] Example: if $f(x) \propto e^{-x^2/2\sigma^2}$ with $\sigma^2 > 0$, then $f^{\rm FT}(p) \propto e^{-\sigma^2 p^2/2}$ in the limit $\epsilon \to 0$ (Hunter (2005), equation 11.24).

[69] Article 30983

28 Pre-smeared composite operators

If the smeared field operator $\phi(f)$ satisfies the average energy test and norm test, then so does any pre-smeared[70] composite operator constructed from $\phi(f)$. This section confirms this directly for one example of such an operator.

Consider the free scalar field again, and let $\phi(f)$ be the field operator smeared only in space. Choose a real number c and consider the pre-smeared operator $\mathcal{O} = e^{ic\phi(f)}$. This operator is unitary, so it clearly passes the norm test. The result in section 27 suggests that this operator will also pass the average energy test. To check this, start with

$$
\begin{aligned}
-i\dot{\mathcal{O}} = [H, \mathcal{O}] &= \frac{1}{2}\int_{\mathbf{x}} \left[\dot{\phi}^2(\mathbf{x}), e^{ic\phi(f)}\right] \\
&= \frac{1}{2}\int_{\mathbf{x}} \Big(\dot{\phi}(\mathbf{x})\big[\dot{\phi}(\mathbf{x}), e^{ic\phi(f)}\big] + \big[\dot{\phi}(\mathbf{x}), e^{ic\phi(f)}\big]\dot{\phi}(\mathbf{x})\Big) \\
&= \frac{c}{2}\Big(\dot{\phi}(f)\mathcal{O} + \mathcal{O}\dot{\phi}(f)\Big) \\
&= \frac{c}{2}[\dot{\phi}(f), \mathcal{O}] + c\,\mathcal{O}\dot{\phi}(f) \\
&= \frac{c^2}{2}\left(\int_{\mathbf{x}} |f(\mathbf{x})|^2\right)\mathcal{O} + c\,\mathcal{O}\dot{\phi}(f)
\end{aligned}
$$

Use that expression for $\dot{\mathcal{O}}$ in

$$
\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\big\langle 0\big|\mathcal{O}^\dagger[H, \mathcal{O}]\big|0\big\rangle}{2\langle\psi|\psi\rangle} = -i\frac{\big\langle 0\big|\mathcal{O}^\dagger\dot{\mathcal{O}}\big|0\big\rangle}{2\big\langle 0\big|\mathcal{O}^\dagger\mathcal{O}\big|0\big\rangle} \tag{31}
$$

and then use $\mathcal{O}^\dagger\mathcal{O} = 1$ and (16) to deduce that the ratio (31) is equal to $\frac{c^2}{4}\int_{\mathbf{x}} |f(\mathbf{x})|^2$. This shows that the pre-smeared operator passes the average energy test, as expected.

---

[70] Section 5

29 Interacting field smeared in space

Sections 27 showed that smearing only in space is enough to make the free scalar field operator pass the norm test and the average energy test. This section shows that smearing only in space is still enough when the field is self-interacting.[71]

First consider the norm test. The **Källén-Lehmann representation** of the 2-point function is[72]

$$\langle 0|\phi(\mathbf{x},t)\phi(\mathbf{y},t)|0\rangle = \int_{\mathbf{p}} \int_0^\infty dp_0 \int_0^\infty ds\ \rho(s)\delta(p_0^2 - \mathbf{p}^2 - s)e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \tag{32}$$

where $\rho(s)$ is a non-negative function satisfying[73,74]

$$\int_0^\infty ds\ \rho(s) = 1. \tag{33}$$

This gives[75]

$$\begin{aligned}\langle 0|\phi^2(f,t)|0\rangle &= \int_{\mathbf{x},\mathbf{y}} f(\mathbf{x})f(\mathbf{y})\langle 0|\phi(\mathbf{x},t)\phi(\mathbf{y},t)|0\rangle \\ &= \int_{\mathbf{p}} \int_0^\infty dp_0 \int_0^\infty ds\ \rho(s)\delta(p_0^2 - \mathbf{p}^2 - s)|f^{\text{FT}}(\mathbf{p})|^2 \\ &= \int_{\mathbf{p}} \int_0^\infty ds\ \rho(s)\frac{|f^{\text{FT}}(\mathbf{p})|^2}{\sqrt{\mathbf{p}^2 + s}}\end{aligned} \tag{34}$$

[71] This result is not anticipated by the intuition in section 2 because this derivation does not assume that $\phi|0\rangle$ is a single-particle state.

[72] Weinberg (1995), equation (10.7.5)

[73] Weinberg (1995), equation (10.7.18)

[74] A representation of the form (32) would still hold if the field operator ϕ were replaced by a composite operator, but the normalization condition (33) is a special property of the field operator ϕ (Weinberg (1995), text above equation (10.7.17)). For other operators, $\rho(s)$ is only "polynomially bounded in the large momentum limit" (Buoninfante *et al* (2024), text after equation (2.3)), which amounts to assuming that smearing in spacetime works (sections 10 and 12).

[75] The square root in the denominator of the integrand comes from evaluating the integral over p_0, because the delta distribution in the integrand may also be written $\delta(p_0^2 - \mathbf{p}^2 - s) = \delta\big((p_0 + (\mathbf{p}^2 + s)^{1/2})(p_0 - (\mathbf{p}^2 + s)^{1/2})\big)$.

This would be finite without the factor $\sqrt{\mathbf{p}^2+s}$ in the denominator, and that factor only makes the integrand decrease even faster for large $\mathbf{p}^2$ and large s. If the single-particle mass m is nonzero, then $\rho(s)=0$ for $s<m^2$, so the minimum value of $\sqrt{\mathbf{p}^2+s}$ is nonzero. Altogether, this shows that the quantity $\langle 0|\phi^2(f,t)|0\rangle$ is finite in smooth space. Use this in (35) to conclude that smearing only in space is enough to make the interacting scalar field pass the norm test.

Now consider the average energy test. If $|\psi\rangle=\phi(f,t)|0\rangle$, then the quantity (27) is

$$\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}=i\frac{\langle 0|[\dot{\phi}(f,t),\phi(f,t)]|0\rangle}{2\langle 0|\phi^2(f,t)|0\rangle}\propto\frac{\int_{\mathbf{x}}\ |f(x)|^2}{\langle 0|\phi^2(f,t)|0\rangle}. \tag{35}$$

The numerator is just like it was for the free scalar field, because the interaction doesn't change the equal-time commutation relation or the part of H that affects its commutator with $\phi(f,t)$.[76] The denominator is different, but the previous paragraph showed that it's not zero and does not approach zero in the smooth-space limit, so the quantity (35) has a finite smooth-space limit. This shows that the operator $\phi(f,t)$ passes the average energy test.

We can also use equation (35) to confirm again that $\phi(f,t)$ passes the norm test. The quantity (35) cannot be zero because the vacuum state $|0\rangle$ is the only state for which the expectation value of H is zero.[77] The numerator in (35) is manifestly finite, so the fact that the ratio is nonzero implies that the denominator must also be finite.

This shows that for the field operator itself, smearing only in space is sufficient.[78] Other examples will show that for most (composite) operators, smearing only in space is not effective, not even for a free scalar field.

[76] H may be expressed in terms of the field operators $\phi(\mathbf{x},t)$ at any given time t, and since H is independent of time, the expression is the same no matter which t we choose (article 52890). The statement about the commutator $[H,\phi(f,t)]$ assumes that the value of t used in the expression for H is the same as the one in $\phi(f,t)$.

[77] This assumes that the $\phi\to-\phi$ symmetry is not spontaneously broken (section 20).

[78] Witten (2023), text after equation (2.2)

30 Renormalization with free fields

The remaining examples will use *composite operators*. They are constructed from the basic field operators $\phi(\mathbf{x},t)$ but that are not linear in those field operators. Correlation functions involving point-localized composite operators are usually not well-behaved, but they are well-behaved when they involve only a subset of those operators called *renormalized* composite operators.[79] This section uses a free scalar field to illustrate those statements.

Even if the field is not free (if V involves higher powers of ϕ), we can always write the field in the form (17) where $\phi^\pm$ are the energy raising/lowering parts of ϕ.[80] The free scalar field is special, though, because in this case the commutator $[\phi^-(x),\phi^+(y)]$ is proportional to the identity operator.[81] This section explains how that special feature can be used to construct renormalized composite operators in the free scalar field model. This will be done in two steps. The first step defines *normal ordering* and shows that the normal-ordered counterpart of a point-localized operator is another operator localized at the same point. The second step shows that correlation functions of these operators are well-behaved.

To define normal ordering, write $\phi(x) = \phi^+(x) + \phi^-(x)$ as in equation (17). Let P be a product with any number of factors of ϕ^+s and any number of factors of ϕ^-, multiplied in any order. Define $(P)_R$ to be the product with those same numbers of factors but with all ϕ^+s on the left and all ϕ^-s on the right.[82] This recipe for modifying P is called **normal ordering**.[83] The subscript R is used because if the field is free, then $(P)_R$ qualifies as a renormalized operator, which will be

[79] This is the defining property of renormalized operators (section 16). The smooth-space limit of a correlation function of point-localized renormalized operators is allowed diverge only when two or more of the points coincide. Those divergences are cured by smearing, not by renormalization.

[80] The energy raising/lowering parts are usually called the creation/annihilation parts. For a *free* scalar field, that language makes sense, because in that model they create/annihilate individual particles. In a model with interactions, the relationship to particles is usually not so simple, but the *energy raising/lowering* language is still accurate.

[81] If the field is not free then $[\phi^-(x),\phi^+(y)]$ is not necessarily proportional to the identity operator, so renormalization requires other methods (Bostelmann and Fewster (2009), end of section 1).

[82] The traditional notation for $(P)_R$ is $:P:$.

[83] Article 23277

demonstrated later in this section. For any two products A and B and coefficients c_A and c_B, the definition is extended to $c_A A + c_B B$ by linearity: $(c_A A + c_B B)_R \equiv c_A (A)_R + c_B (B)_R$.

The first goal is to show that the normal-ordered counterpart of a point-localized operator is another operator localized at the same point. For this, consider an example. Start with the operator $\phi^4(x)$. This operator is well-defined when space is discrete. Write it as $\phi^4(x) = \big(\phi^+(x) + \phi^-(x)\big)^4$ to get a sum of products of the energy raising/lowering operators $\phi^\pm(x)$ in various orders. Use the trivial identity

$$\phi^-(x)\phi^+(y) = \phi^+(y)\phi^-(x) + [\phi^-(x), \phi^+(y)] \tag{36}$$

and the fact that $[\phi^-(x), \phi^+(y)]$ is proportional to the identity operator to get

$$\phi^4(x) = (\phi^4(x))_R + c_2\,(\phi^2(x))_R + c_0 \tag{37}$$

where the coefficients c_k are functions of the ϵ-dependent real number $[\phi^-(x), \phi^+(x)]$. Similarly, for any positive integer n, the composite operator $\phi^n(x)$ may be written as a linear combination of normal ordered operators $(\phi^k(x))_R$ with $0 \leq k \leq n$. This relationship between the operators $\phi^n(x)$ and their normal-ordered counterparts may be written

$$\phi^n(x) = \sum_{k\leq n} M_{n,k}\,(\phi^k(x))_R \tag{38}$$

with real-valued coefficients $M_{n,k}$. This implies[84]

$$(\phi^n(x))_R = \sum_{k\leq n} (M^{-1})_{n,k}\,\phi^k(x) \tag{39}$$

where M^{-1} is the inverse of the matrix M. This shows that $(\phi^n(x))_R$ is polynomial in $\phi(x)$ with order n. The operator $\phi(x)$ is localized at x by definition, so this shows that $(\phi^n(x))_R$ is also localized at x.

[84] Proof: write $M = M_0 + M_1$ where M_0 is diagonal and $(M_1)_{n,k} = 0$ whenever $k \geq n$. Then $M_1^N = 0$ for sufficiently large N. Write $M = M_0(I + M_0^{-1}M_1)$ to get

$$M^{-1} = (I + M_0^{-1}M_1)^{-1}M_0^{-1} = (I - M_0^{-1}M_1 + (M_0^{-1}M_1)^2 - \cdots \text{(up to } N\text{th power)})M_0^{-1}.$$

The next goal is to show that correlation functions of the operators $(\phi^n(x))_R$ are well-behaved. For this, consider another example, the correlation function of two such operators:

$$\big\langle 0\big|(\phi^n(x))_R(\phi^k(y))_R\big|0\big\rangle. \tag{40}$$

Use the identity (36) together with $\phi^-|0\rangle = 0$ and the fact that $[\phi^-(x), \phi^+(y)]$ is proportional to the identity operator to reduce this 2-point function to a polynomial in the quantity $[\phi^-(x), \phi^+(y)]$. That quantity is well-behaved in the sense that it is finite for $x \neq y$ and nonzero in a neighborhood of $x = y$ whose size remains nonzero in the smooth-space limit. This shows that the 2-point function (40) is also well-behaved in this sense. The same reasoning works for any correlation function of normal-ordered operators.

Altogether, this shows that $(\phi^n(x))_R$ qualifies as a renormalized counterpart of $\phi^n(x)$. By the way, this also shows that correlation functions like

$$\big\langle 0\big|\phi^n(x)\phi^k(y)\big|0\big\rangle \tag{41}$$

are typically not well-behaved (unless $n = k = 1$). To deduce this, use equation (38) and the fact that the coefficients in that equation are proportional to powers of the quantity $[\phi^-(x), \phi^+(x)] \sim 1/\epsilon^{D-1}$ (if $D \geq 2$),[85] which diverges when $\epsilon \to 0$.

Any linear combination of renormalized operators with ϵ-independent coefficients is still a renormalized operator (still has well-behaved correlation functions). That implies that renormalization is not unique: equation (39) is not the only nth-order polynomial in $\phi(x)$ that qualifies as a renormalized operator.

Beware that in the physics literature, unrenormalized operators and their renormalized counterparts of are often denoted using the same symbol, without any decorations (like the subscript R) to distinguish them from each other. This tendency is strangely more prevalent in the context of models with interactions, even though constructing renormalized composite operators is (much) more difficult that it is when interactions are absent.

[85] Section 24

31 $(\phi^2)_R$ smeared in space: norm test

Let $\phi(x)$ be a free scalar field in D-dimensional space, and consider the normal-ordered version of $\phi^2(x)$, which is[86]

$$(\phi^2(x))_R = \phi^2(x) - \langle 0|\phi^2(x)|0\rangle. \tag{42}$$

Smearing this operator in space gives an operator

$$\mathcal{O} \equiv \int_{\mathbf{x}} f(\mathbf{x})(\phi^2(\mathbf{x}, t=0))_R. \tag{43}$$

When space is discrete, this is a well-defined operator on the Hilbert space, but it becomes ill-defined in the smooth-space limit if the number of dimensions of space is $D \geq 2$. This section shows that $\mathcal{O}$ fails the norm test, and section 33 will show that it also fails the average energy test.

Use the expression for $\phi(\mathbf{x})$ from section 21 to get

$$(\phi^2(\mathbf{x}))_R|0\rangle \propto \int_{\mathbf{p}_1,\mathbf{p}_2} \frac{a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)e^{-i(\mathbf{p}_1+\mathbf{p}_2)\cdot\mathbf{x}}}{\sqrt{\omega(\mathbf{p}_1)\omega(\mathbf{p}_2)}}|0\rangle, \tag{44}$$

with the understanding that the integrals are cut off when the magnitude of any momentum component reaches π/ϵ because we haven't taken the smooth-space limit yet. Define $G_2(\mathbf{x}-\mathbf{x}') \equiv \langle 0|(\phi^2(\mathbf{x}'))_R\,(\phi^2(\mathbf{x}))_R|0\rangle$ and use equations (19) to get

$$G_2(\mathbf{x}-\mathbf{x}') \propto \big(g(\mathbf{x}-\mathbf{x}')\big)^2 \tag{45}$$

with g defined by (23). The operator $\mathcal{O}$ is defined by smearing $(\phi^2(\mathbf{x}))_R$ in space before taking the smooth-space limit. To check whether $\mathcal{O}$ passes the norm test, we should consider the smooth-spacetime limit of the quantity

$$\langle 0|\mathcal{O}\mathcal{O}|0\rangle = \int_{\mathbf{x},\mathbf{x}'} f(\mathbf{x})f(\mathbf{x}')G_2(\mathbf{x}-\mathbf{x}'). \tag{46}$$

[86] Section 30

To make this more explicit, use the smearing function $f(\mathbf{x}) \propto e^{-\mathbf{x}^2/2\sigma^2}$ and evaluate the integrals over $\mathbf{x}$ and $\mathbf{x}'$ first to get[87]

$$\langle 0|\mathcal{O}\mathcal{O}|0\rangle \propto \int_{\mathbf{p}_1,\mathbf{p}_2} \frac{e^{-(\mathbf{p}_1+\mathbf{p}_2)^2\sigma^2}}{\omega(\mathbf{p}_1)\omega(\mathbf{p}_2)}. \tag{47}$$

To determine how this depends on ϵ, use the abbreviations $\mathbf{p}_\pm \equiv \mathbf{p}_1 \pm \mathbf{p}_2$. The exponential factor in the integrand effectively prevents the magnitude of $\mathbf{p}_+$ from being much greater than $1/\sigma$, but the magnitude of $\mathbf{p}_-$ can still be $\sim 1/\epsilon$, so the factors of ω in the denominator can both still be $\sim 1/\epsilon$. This gives

$$\langle 0|\mathcal{O}\mathcal{O}|0\rangle \overset{\propto}{\sim} \frac{1}{\sigma^D \epsilon^{D-2}} \qquad\qquad \text{if } D \geq 2. \tag{48}$$

The case $D = 1$ will not be considered here. It requires special handling because if $m = 0$ then the integrand would diverge as $\mathbf{p}_k \to \mathbf{0}$, so the dependence on m cannot be ignored in that case.

This shows that if the number of dimensions of space is $D \geq 2$ (that is, if the number of dimensions of spacetime $d \geq 3$), then the operator $\mathcal{O}$ fails the norm test: the norm of $\mathcal{O}|0\rangle$ becomes undefined in the smooth-space limit.

The message is that smearing the point-localized operator (42) only in space is insufficient. Section 34 will show that this is typical: (post-)smearing only in space is almost never sufficient, even though it is sufficient for the field operator itself.[88]

[87]Switching the order of integration is allowed because space is discrete and finite, so the "integrals" are really finite sums. The smooth-space limit or infinite-volume limits are taken only after obtaining expressions that are well-defined in those limits.

[88]Sections 27-29

32 Another method

In section 31, the quantity (46) was studied by evaluating the integrals over $\mathbf{x}, \mathbf{x}'$ before the integrals over $\mathbf{p}, \mathbf{p}'$. This section confirms that integrating over $\mathbf{p}, \mathbf{p}'$ first leads to the same conclusion.

For $|\mathbf{x}| \ll 1/m$, the smooth-space limit of the function (45) would be[89]

$$G_2(\mathbf{x}) \sim \left(\frac{1}{|\mathbf{x}|^{D-1}}\right)^2,$$

but before the smooth-space limit, equation (25) says it approaches $\propto (1/\epsilon^{D-1})^2$ when $|\mathbf{x}| \to 0$. Since the goal is only to determine how $\langle 0|\mathcal{O}^2|0\rangle$ depends on ϵ, we can use these properties of $G_2(\mathbf{x})$ and otherwise treat the sums over lattice points as smooth integrals (like the notation already suggests). Using the abbreviations $\mathbf{x}_\pm \equiv \mathbf{x} \pm \mathbf{x}'$ and the identity

$$e^{-\mathbf{x}^2/2\sigma^2} e^{-(\mathbf{x}')^2/2\sigma^2} = e^{-\mathbf{x}_+^2/4\sigma^2} e^{-\mathbf{x}_-^2/4\sigma^2},$$

this gives

$$\begin{aligned}\int_{\mathbf{x},\mathbf{x}'} f(\mathbf{x}) f(\mathbf{x}') G_2(\mathbf{x}_-) &\propto \frac{1}{\sigma^{2D}} \int_{\mathbf{x},\mathbf{x}'} e^{-\mathbf{x}_+^2/4\sigma^2} e^{-\mathbf{x}_-^2/4\sigma^2} G_2(\mathbf{x}_-) \\ &\propto \frac{1}{\sigma^D} \int_{\mathbf{x}_-} e^{-\mathbf{x}_-^2/4\sigma^2} G_2(\mathbf{x}_-),\end{aligned}$$

and then equation (25) gives

$$\begin{aligned}\int_{\mathbf{x},\mathbf{x}'} f(\mathbf{x}) f(\mathbf{x}') G_2(\mathbf{x}_-) &\propto \frac{\epsilon^D}{\sigma^D} G_2(\mathbf{0}) \quad \text{if } D \geq 2 \\ &\propto \frac{1}{\sigma^D \epsilon^{D-2}} \quad \text{if } D \geq 2.\end{aligned}$$

This agrees with the previous result (48).

[89]Equations (24) and (45)

33 $(\phi^2)_R$ smeared in space: average energy test

Section 31 showed that the operator (43) fails the norm test. This section shows that it also fails the average energy test.

Section 31 already evaluated $\langle\psi|\psi\rangle$. The quantity $\langle\psi|H|\psi\rangle$ in the numerator of the average energy test can be evaluated in a similar way. The result is like (47) but with a new factor of $\omega(\mathbf{p}_1)+\omega(\mathbf{p}_2)$ in the integrand:

$$\langle 0|\mathcal{O}H\mathcal{O}|0\rangle \propto \int_{\mathbf{p}_1,\mathbf{p}_2} \frac{\omega(\mathbf{p}_1)+\omega(\mathbf{p}_2)}{\omega(\mathbf{p}_1)\omega(\mathbf{p}_2)} e^{-(\mathbf{p}_1+\mathbf{p}_2)^2\sigma^2/2}, \tag{49}$$

so

$$\langle 0|\mathcal{O}H\mathcal{O}|0\rangle \underset{\sim}{\propto} \frac{1}{\sigma^D\epsilon^{D-1}}. \tag{50}$$

Combine this with the previous result (48) to get

$$\frac{\langle 0|\mathcal{O}H\mathcal{O}|0\rangle}{\langle 0|\mathcal{O}\mathcal{O}|0\rangle} \underset{\sim}{\propto} \frac{1}{\epsilon}.$$

This diverges in the smooth-space limit $\epsilon\to 0$, so $\mathcal{O}$ fails the average energy test.

34 $(\phi^n)_R$ smeared in space

The analysis in sections 31-33 can be generalized to $(\phi^n(\mathbf{x}))_R$ for any positive integer n. This section shows that for every $n \geq 2$, the operator

$$\mathcal{O} \equiv \int_{\mathbf{x}} f(\mathbf{x})(\phi^n(\mathbf{x}, t=0))_R \tag{51}$$

fails the norm test and the average energy test. This illustrates the claim that smearing a point-localized operator only in space is almost always insufficient.

First consider the norm test. Use the fact that the only term in $(\phi^n(\mathbf{x}))_R$ that contributes to $(\phi^n(\mathbf{x}))_R|0\rangle$ is the term involving only energy-increasing operators $a^\dagger(\mathbf{p})$, not the energy-decreasing operators $a(\mathbf{p})$. This generalizes equation (44) to arbitrary n. That leads immediately to this generalization of (45):

$$G_n(\mathbf{x}-\mathbf{x}') \equiv \langle 0|(\phi^n(\mathbf{x}'))_R\,(\phi^n(\mathbf{x}))_R|0\rangle \propto \big(g(\mathbf{x}-\mathbf{x}')\big)^n \tag{52}$$

Proceed like in section 32 to get

$$\begin{aligned}\langle 0|\mathcal{O}\mathcal{O}|0\rangle &= \int_{\mathbf{x},\mathbf{x}'} f(\mathbf{x})f(\mathbf{x}')G_n(\mathbf{x}_-)\\ &\propto \frac{\epsilon^D}{\sigma^D}\,G_n(\mathbf{0}) \;\;\text{if } D\geq 2\\ &\propto \frac{\epsilon^D}{\sigma^D\epsilon^{n(D-1)}} \;\;\text{if } D\geq 2.\end{aligned}$$

This shows that the operator (51) fails the norm test for every $n \geq 2$.

Generalizing the analysis in section 33 to any n shows that it also fails the average energy test.

35 Conformal field theory

The result in section 34 can be generalized to any model that would, in the infinite-volume and smooth-space limits, be scale-invariant. The study of scale-invariant models is called **conformal field theory**. For such a model, spatial smearing is sufficient for the norm test if and only if the dimension Δ of an operator satisfies $2\Delta < D$.[90] To deduce this, use

$$G(\mathbf{x}-\mathbf{y}) \equiv \langle 0|\mathcal{O}(\mathbf{x})\mathcal{O}(\mathbf{y})|0\rangle \propto |\mathbf{x}-\mathbf{y}|^{-2\Delta}, \tag{53}$$

which follows from the definition of Δ. If the smearing function is $f(\mathbf{x}) \propto e^{-\mathbf{x}^2/2\sigma^2}$, then an approach like the one in section 32 gives

$$\int d^D x'\, d^D x\; f(\mathbf{x})f(\mathbf{x}')G(\mathbf{x}'-\mathbf{x}) \sim \frac{\epsilon^D}{\sigma^D \epsilon^{-2\Delta}}. \tag{54}$$

For a free scalar field, we can use naïve dimensional analysis $[\phi^n] = [m]^{(D-1)n/2}$ to get $2\Delta = n(D-1)$, and then this reproduces the analysis in section 34.

[90] Witten (2023), text after equation (2.2)

36 $(\phi^n)_R$ smeared in time: norm test

This section shows that if ϕ is a free scalar field and n is a positive integer, then the operator defined by smearing $(\phi^n(\mathbf{x},t))_R$ in time passes the norm test.[91,92]

Setting $\mathbf{x}=\mathbf{0}$ and smearing in time gives the operator

$$\mathcal{O} \equiv \int dt\; f(t)\,(\phi^n(\mathbf{x},t))_R. \tag{55}$$

Using the abbreviation $\omega_k \equiv \omega(\mathbf{p}_k)$, the norm of $\mathcal{O}|0\rangle$ is

$$\begin{aligned}\langle 0|\mathcal{O}^2|0\rangle &= \int dt\,dt'\; f(t)f(t')\langle 0|\big(\phi^-(\mathbf{0},t')\big)^n\big(\phi^+(\mathbf{0},t)\big)^n|0\rangle \\ &\propto \int dt\,dt'\; f(t)f(t')\left(\int_{\mathbf{p}}\frac{e^{i\omega(\mathbf{p})(t-t')}}{\omega(\mathbf{p})}\right)^n \\ &= \int dt\,dt'\; f(t)f(t')\int_{\mathbf{p}_1,\ldots,\mathbf{p}_n}\frac{e^{i(\omega_1+\cdots+\omega_n)(t-t')}}{\omega_1\cdots\omega_n}\end{aligned} \tag{56}$$

with a proportionality factor that remains finite as $\epsilon\to 0$. Use the smearing function $f(t)\propto e^{-t^2/2\sigma^2}$ to get

$$\langle 0|\mathcal{O}^2|0\rangle \propto \int_{\mathbf{p}_1,\ldots,\mathbf{p}_n}\frac{e^{-(\omega_1+\cdots+\omega_n)^2\sigma^2}}{\omega_1\cdots\omega_n}. \tag{57}$$

The exponential factor in the integrand suppresses large values of the quantity $\omega_1+\cdots+\omega_n$. Ignoring the cross-terms $\omega_j\omega_k$ with $j\neq k$ in the exponential reduces

[91] $(\cdots)_R$ denotes a normal-ordered operator.

[92] Equation (2.4) in Fewster and Verch (2013) shows the result of a more general calculation in which the vacuum state $|0\rangle$ is replaced by a more general family of states. Instead of using the norm test to assess the health of the operator, they use it to characterize the state (theorem 2.3).

the degree of suppression. Use this fact together with $\sigma \ll 1/m$ to get[93,94]

$$\int_{\mathbf{p}_1,\ldots,\mathbf{p}_n} \frac{e^{-(\omega_1+\cdots+\omega_n)^2\sigma^2}}{\omega_1\cdots\omega_n} \leq \left(\int_{\mathbf{p}} \frac{e^{-\omega^2\sigma^2}}{\omega}\right)^n \underset{\sim}{\propto} \left(\frac{1}{\sigma^{D-1}}\right)^n, \tag{58}$$

which shows that the operator (55) passes the norm test.

[93] If $\sigma \ll 1/m$, then the integral is dominated by momenta with $|\mathbf{p}| \gg m$, so $\omega \approx |\mathbf{p}|$.
[94] Footnote 101 in section 38 also applies here.

37 Another method

In section 36, the quantity (56) was studied by evaluating the time integrals before the momentum integrals. This section confirms that doing the momentum integrals first leads to the came conclusion.[95] This approach will take more effort, but it is included here because section 39 will use a similar approach to explore what would happen if the algebraic product $\mathcal{O}^2$ is were replaced by a time-ordered product.

Remember that space is being treated as a finite lattice so that the integrals are really finite sums. Taking the infinite-volume limit is not a problem: the sum over momenta becomes an integral over momentum, as the notation suggests. In contrast, the smooth-space limit cannot be taken before the time integrals are evaluated, because then the momentum integrals would be undefined.[96]

To evaluate the momentum integrals first, we need to account for the fact that when space is discrete, the domain of the momentum integrals is compact. Since $\omega(\mathbf{p})$ is positive for all $\mathbf{p}$, we can achieve the same effect by replacing

$$\int_{\mathbf{p}} \frac{e^{i\omega(\mathbf{p})t}}{\omega(\mathbf{p})} \to \int_{\mathbf{p}} \frac{e^{i\omega(\mathbf{p})(t+i\hat{\epsilon})}}{\omega(\mathbf{p})}$$

with $\hat{\epsilon} \sim \epsilon$. This makes the integrand absolutely integrable with no restriction on the integration domain, so now we can allow the momentum integrals to extend to arbitrarily large momenta. Now we can use the condition $\epsilon \ll 1/m$ to justify neglecting the dependence on m (so $\omega \approx |\mathbf{p}|$) and then use dimensional analysis to infer[97]

$$\int_{\mathbf{p}} \frac{e^{i\omega(\mathbf{p})(t+i\hat{\epsilon})}}{\omega(\mathbf{p})} \approx \int_0^\infty d\omega\ \frac{\omega^{D-1}\, e^{i\omega\,(t+i\hat{\epsilon})}}{\omega} \mathrel{\substack{\propto\\ \sim}} \frac{1}{(t+i\hat{\epsilon})^{D-1}} \qquad \text{for } t \ll \frac{1}{m}. \tag{59}$$

[95]The relationship between this section and section 36 analogous to the relationship between sections 32 and (31), but for smearing in time instead of smearing in space.

[96]When space is discrete, the domain of integration for the momentum integrals is finite, but in the smooth-space limit the momentum integration variables can be arbitrarily large. In the naïve smooth-space limit, rotational symmetry would give $\int_{\mathbf{p}} \frac{e^{i\omega t}}{\omega} \propto \int_0^\infty dp\ \frac{p^{D-1}}{\omega} e^{i\omega t}$, so the integrand is not absolutely integrable over $0 < p < \infty$.

[97]Without the $i\hat{\epsilon}$ term, dimensional analysis would give $\int_{\mathbf{p}} \frac{\exp(i\omega(\mathbf{p})t)}{\omega(\mathbf{p})} \mathrel{\substack{\propto\\ \sim}} 1/t^{D-1}$ only for t in the range $\epsilon \ll t \ll 1/m$, where the integral is approximately independent of both the lattice spacing ϵ and the mass m.

The $i\hat{\epsilon}$ term in the denominator of equation (59) is essential, so let's check it another way. Define a function $\theta(\omega)$ by

$$\theta(\omega) \equiv \begin{cases} 1 & \text{if } \omega > 0 \\ 0 & \text{if } \omega < 0, \end{cases}$$

and write $k \equiv D - 1$. Then equation (59) says that for any positive integer $k \geq 1$, the Fourier transform of $\theta(\omega)\omega^{k-1}$ is $1/(t + i\hat{\epsilon})^k$. We can check this by calculating the inverse Fourier transform

$$\int dt \; \frac{e^{-i\omega t}}{(t + i\hat{\epsilon})^k}.$$

If $\omega > 0$, then the integration contour can be closed by looping back around the lower half of the complex plane. Then the contour encloses the pole at $t = -i\hat{\epsilon}$, so Cauchy's integral theorem says the integral is proportional to ω^{k-1} in that case. If $\omega < 0$, then the integration contour can be closed in the upper half of the complex plane. Then the contour does not enclose a pole, so Cauchy's integral theorem says the integral is zero in that case. This gives[98]

$$\int dt \; \frac{e^{-i\omega t}}{(t + i\hat{\epsilon})^k} \propto \theta(\omega)\omega^{k-1},$$

which is consistent with (59).

Now return to the task of evaluating (56). Using (59) to evaluate the momentum integrals in (56) gives

$$\begin{aligned} \langle 0|\mathcal{O}^2|0\rangle &\propto \int dt\, dt' \; f(t)f(t') \left(\frac{1}{(t - t' + i\hat{\epsilon})^{D-1}} \right)^n \\ &= \int dt\, dt' \; f(t)f(t') \frac{1}{(t - t' + i\hat{\epsilon})^k} \qquad \text{with } k \equiv n(D-1). \end{aligned} \tag{60}$$

[98]Switching the roles of t and ω gives a relationship that many introductions to quantum field theory use to relate time-ordered correlation functions – which are defined using factors of $\theta(\pm t)$ (section 39) – to propagators that have a small imaginary offset in the momentum domain. That small imaginary offset is sometimes called *Feynman's prescription* (example: Visser (2022)).

As before, use the smearing function $f(t) \propto e^{-t^2/2\sigma^2}$ with $\sigma \ll 1/m$. To show that (60) is finite whenever $D \geq 2$, use the identity

$$\left(\frac{d}{dt}\right)^{k+1} \big((\log t - 1)t\big) \propto \frac{1}{t^k},$$

which is true for any positive integer k. Use this in (60) and then integrate by parts[99] to get

$$\langle 0|\mathcal{O}^2|0\rangle \propto \int dt\, dt'\, \big(\log(t - t' + i\hat{\epsilon}) - 1\big)(t - t' + i\hat{\epsilon}) \left(\frac{d}{dt}\right)^{k+1} f(t)f(t').$$

This is manifestly finite even when $\hat{\epsilon} = 0$, so now we can set $\hat{\epsilon} = 0$ (which corresponds to taking the smooth-space limit in (56)) and use dimensional analysis to get

$$\langle 0|\mathcal{O}^2|0\rangle \propto \frac{1}{\sigma^k},$$

which agree with the previous result (58),[100] showing once again that the operator $\mathcal{O}$ defined in (55) passes the norm test.

[99] Witten (2023), equations (2.3)-(2.5)

[100] Recall that equation (60) defined $k \equiv n(D-1)$.

38 $(\phi^n)_R$ smeared in time: average energy test

Consider the operator $\mathcal{O}$ defined by (55), and use the abbreviation $|\psi\rangle \equiv \mathcal{O}|0\rangle$. Section 36 already evaluated $\langle\psi|\psi\rangle$. The quantity $\langle\psi|H|\psi\rangle$ in the numerator of the average energy test can be evaluated in a similar way. This gives an integral like (57) but with a new factor of $\omega_1 + \cdots + \omega_n$ in the integrand:

$$\langle 0|\mathcal{O}H\mathcal{O}|0\rangle \propto \int_{\mathbf{p}_1,\ldots,\mathbf{p}_n} \frac{\omega_1 + \cdots + \omega_n}{\omega_1 \cdots \omega_n} e^{-(\omega_1+\cdots+\omega_n)^2\sigma^2}. \tag{61}$$

Just like in section 36, this gives

$$\begin{aligned}\int_{\mathbf{p}_1,\ldots,\mathbf{p}_n} \frac{\omega_1 + \cdots + \omega_n}{\omega_1 \cdots \omega_n}\, e^{-(\omega_1+\cdots+\omega_n)^2\sigma^2} &\leq \left(\int_{\mathbf{p}} \frac{e^{-\omega^2\sigma^2}}{\omega}\right)^{n-1} \int_{\mathbf{p}} e^{-\omega^2\sigma^2} \\ &\underset{\sim}{\propto} \left(\frac{1}{\sigma^{D-1}}\right)^{n-1} \frac{1}{\sigma^D}.\end{aligned} \tag{62}$$

For narrow smearing widths σ, the integrals (57) and (61) are both dominated by parts of the integration domain where at least one ω_k is large, so ignoring the cross-terms $\omega_j\omega_k$ with $j \neq k$ makes the value of the ratio $\langle 0|\mathcal{O}H\mathcal{O}|0\rangle/\langle 0|\mathcal{O}\mathcal{O}|0\rangle$ larger.[101] Use this with equations (58) and (62) to get

$$\frac{\langle 0|\mathcal{O}H\mathcal{O}|0\rangle}{\langle 0|\mathcal{O}\mathcal{O}|0\rangle} \leq \left.\frac{\langle 0|\mathcal{O}H\mathcal{O}|0\rangle}{\langle 0|\mathcal{O}\mathcal{O}|0\rangle}\right|_{\text{no cross terms}} \underset{\sim}{\propto} \frac{1}{\sigma} \ll \frac{1}{\epsilon}.$$

This shows that $\mathcal{O}$ passes the average energy test.

[101] Instead of using this reasoning about the cross-terms, we can simply use the fact that if $\epsilon \ll \sigma \ll 1/m$, then the dependence on ϵ and $1/m$ is negligible, and then the result follows from dimensional analysis. The cross-terms argument is introduced here because it will be used again in section 44, where dimensional analysis is not sufficient because the value of the proportionality factor will be important (even though it remains finite as $\epsilon \to 0$).

39 Time-ordered products and time-smearing

Section 15 mentioned that the results in this article about the effectiveness of smearing in time apply only when restricted to algebraic products of the smeared operators, not when using time-ordered products. Sections 36-37 showed – using two different methods – that the operator defined by smearing $(\phi^n(\mathbf{x},t))_R$ in time passes the norm test, which is possible because the norm test defined in section 17 is expressed using algebraic products. This section shows will show that the same operator would fail the test if the test were modified to use the time-ordered product instead.[102]

Use the abbreviation $\mathcal{O}(t) \equiv (\phi^n(\mathbf{0},t))_R$ for the unsmeared operator. The first line of equation (56) involves the algebraically ordered 2-point function

$$\langle 0|\mathcal{O}(t')\mathcal{O}(t)|0\rangle.$$

Suppose that it were replaced with the time-ordered 2-point function

$$\langle 0|T\mathcal{O}(t')\mathcal{O}(t)|0\rangle \equiv \theta(t'-t)\langle 0|\mathcal{O}(t')\mathcal{O}(t)|0\rangle + \theta(t-t')\langle 0|\mathcal{O}(t)\mathcal{O}(t')|0\rangle.$$

The analog of equation (60) would be

$$\begin{aligned}
\langle 0|T\mathcal{O}^2|0\rangle &\propto \int dt\,dt'\;f(t)f(t')\left(\frac{\theta(t'-t)}{(i\hat\epsilon+t-t')^k}+\frac{\theta(t-t')}{(i\hat\epsilon+t'-t)^k}\right)\\
&= \int dt\,dt'\;f(t+t')f(t')\left(\frac{\theta(-t)}{(i\hat\epsilon+t)^k}+\frac{\theta(t)}{(i\hat\epsilon-t)^k}\right)\\
&= \int dt'\;f(t')\left(\int_{-\infty}^{0}dt\;\frac{f(t+t')}{(i\hat\epsilon+t)^k}+\int_0^\infty dt\;\frac{f(t+t')}{(i\hat\epsilon-t)^k}\right).
\end{aligned} \tag{63}$$

This differs from equation (60) only because of the sign of t in the denominator of the second term, but this sign has an enormous effect. To make the comparison

[102] Using the words *same operator* in this sentence is an example of equivocation, because the definition of the time-ordered product of time-smeared operators uses a generalized definition of *operator*.

more clear, use the abbreviation

$$I_s(t',\epsilon,k) \equiv \int_{-\infty}^{0} dt\ \frac{f(t+t')}{(i\epsilon+t)^k} + \int_0^{\infty} dt\ \frac{f(t+t')}{(i\epsilon+st)^k} \tag{64}$$

with $s=\pm 1$. When the sign is $s=-1$, (64) is the quantity in large parentheses on the last line of (63). When the sign is $s=+1$, (64) gives the corresponding quantity without time-ordering. If $k\geq 2$, then we have the identity

$$(i\epsilon+st)^{-k} = \frac{d}{dt}\frac{(i\epsilon+st)^{1-k}}{(1-k)s}$$

Use this in (64) and integrate by parts to get

$$\begin{aligned} I_s(t',\epsilon,k) &= \frac{1}{1-k}\left(-\int_{-\infty}^{0} dt\ \frac{\dot f(t+t')}{(i\epsilon+t)^{k-1}} - \int_0^{\infty} dt\ \frac{\dot f(t+t')}{(i\epsilon+st)^{k-1}s}\right) \\ &\quad + \frac{1}{1-k}\left(\frac{f(t')}{(i\epsilon)^k} - \frac{f(t')}{(i\epsilon)^k s}\right) \end{aligned} \tag{65}$$

where $\dot f \equiv df/dt$. The quantities on the second line are the boundary terms from integration by parts. When $s=+1$, the boundary terms cancel each other, but when $s=-1$ they don't, so the integral (63) diverges as $\epsilon\to 0$.[103]

Taking $\epsilon\to 0$ corresponds to taking the smooth-space limit, and time-ordered correlation functions in lorentzian spacetime correspond (via Wick rotation) to correlation functions in euclidean spacetime. Thanks to those correspondences, the result derived above shows that "smearing in euclidean time" is not effective for $k\equiv n(D-1)\geq 2$,[104] as promised in section 15.

[103]Dimensional analysis suffices to show that even if the integrals on the first line of (65) diverge as $\epsilon\to 0$, they cannot diverge quickly enough to cancel the divergence on the second line, because the numerator $\dot f(t+t')$ is independent of ϵ.

[104]Equation (60) introduced this definition of k, where n is the exponent in $\mathcal{O}\equiv(\phi^n)_R$ and D is the number of dimensions of space.

40 Exponential operator $e^{ic\phi}$ without smearing

The next series of sections (sections 40-43) illustrates the warning issued in section 18: an operator might be unhealthy even if it passes the norm test. In this example, the "deceptive" normalization factor is already present in the naïve operator instead of being introduced deliberately. This section shows that the naïve operator satisfies the norm test but fails the average energy test.

Let $\phi(\mathbf{x})$ be a scalar field (possibly with self-interaction). Choose a real number c and consider the point-localized operator $\mathcal{O}(\mathbf{x}) \equiv e^{ic\phi(\mathbf{x})}$. This operator is unitary, so it obviously passes the norm test. For the average energy test, use the abbreviation $|\psi\rangle \equiv \mathcal{O}|0\rangle$ and the property $H|0\rangle = 0$[105] to get this identity:

$$\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\big\langle 0\big|\mathcal{O}^\dagger[H,\mathcal{O}]\big|0\big\rangle}{\langle\psi|\psi\rangle}. \tag{66}$$

The hamiltonian has the form (15) where $V[\phi]$ commutes with $\phi(\mathbf{x})$, so the commutation relation (14) gives

$$\begin{aligned}
[H,\mathcal{O}(\mathbf{x})] &= \frac{1}{2}\epsilon^D\sum_{\mathbf{x}'}\Big(\dot\phi(\mathbf{x}')\big[\dot\phi(\mathbf{x}'),\mathcal{O}(\mathbf{x})\big] + \big[\dot\phi(\mathbf{x}'),\mathcal{O}(\mathbf{x})\big]\dot\phi(\mathbf{x}')\Big)\\
&= \frac{c}{2}\big(\dot\phi(\mathbf{x})\mathcal{O}(\mathbf{x}) + \mathcal{O}(\mathbf{x})\dot\phi(\mathbf{x})\big)\\
&= \frac{c}{2}[\dot\phi(\mathbf{x}),\mathcal{O}(\mathbf{x})] + \mathcal{O}(\mathbf{x})\dot\phi(\mathbf{x}) = \frac{c^2}{2\epsilon^D}\mathcal{O}(\mathbf{x}) + \mathcal{O}(\mathbf{x})\dot\phi(\mathbf{x}).
\end{aligned}$$

Use this and (16) in (66) to get

$$\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{c^2}{2\epsilon^D}. \tag{67}$$

This shows that $\mathcal{O}(\mathbf{x})$ fails the average energy test, which is expected because it is localized at a point in spacetime.[106]

[105] Equation (5)

[106] This section used $\phi(\mathbf{x})$ and $\mathcal{O}(\mathbf{x})$ as abbreviations for $\phi(\mathbf{x}, t=0)$ and $\mathcal{O}(\mathbf{x}, t=0)$.

41 Correlation functions of $e^{ic\phi}$

This section shows that correlation functions of the (unrenormalized) exponential operator $e^{ic\phi(\mathbf{x})}$ are not well-behaved.

Let $\phi(\mathbf{x})$ be a free scalar field, choose a real number c, and define

$$\mathcal{O}(\mathbf{x}) \equiv e^{ic\phi(\mathbf{x})}. \tag{68}$$

To evaluate the correlation function $\langle 0|\mathcal{O}^\dagger(\mathbf{y})\mathcal{O}(\mathbf{x})|0\rangle$, let $\phi^\pm(\mathbf{x})$ denote the energy raising/lowering parts of $\phi(\mathbf{x})$.[107] If A and B are operators whose commutator $[A,B]$ is proportional to the identity operator, then[108] $e^Ae^B = e^{A+B+[A,B]/2}$. Use this and equation (22) to get

$$e^{ic\phi^-(\mathbf{x})}e^{ic\phi^+(\mathbf{y})}e^{c^2g(\mathbf{x}-\mathbf{y})/2} = e^{ic(\phi^+(\mathbf{y})+\phi^-(\mathbf{x}))} = e^{ic\phi^+(\mathbf{y})}e^{ic\phi^-(\mathbf{x})}e^{-c^2g(\mathbf{x}-\mathbf{y})/2}. \tag{69}$$

with $g(\mathbf{x})$ defined by (22). A similar manipulation gives

$$e^{ic(\phi(\mathbf{x})-\phi(\mathbf{y}))} = \left(e^{ic(\phi(\mathbf{x})-\phi(\mathbf{y}))^+}\right)\left(e^{ic(\phi(\mathbf{x})-\phi(\mathbf{y}))^-}\right)e^{-2c^2h(\mathbf{x}-\mathbf{y})}$$

with

$$h(\mathbf{x}) \equiv g(\mathbf{0}) - g(\mathbf{x}) \geq 0,$$

so the correlation function is

$$\langle 0|\mathcal{O}^\dagger(\mathbf{y})\mathcal{O}(\mathbf{x})|0\rangle = \langle 0|e^{ic(\phi(\mathbf{x})-\phi(\mathbf{y}))}|0\rangle = e^{-2c^2h(\mathbf{x}-\mathbf{y})}. \tag{70}$$

This is well-defined in discrete space, but something strange happens in the smooth-space limit. For $|\mathbf{x}| \gg \epsilon$, the smooth-space limit of $g(\mathbf{x})$ has the form

$$g(\mathbf{x}) \sim \frac{1}{|\mathbf{x}|^{D-1}},$$

[107] Equation (17)

[108] Article 89053

which is finite for all finite $|\mathbf{x}|$, but the quantity $g(\mathbf{0})$ diverges. As a result, the smooth-space limit of the function $e^{-2c^2 h(\mathbf{x}-\mathbf{y})}$ is zero whenever $\mathbf{x} \neq \mathbf{y}$. In that limit, equation (70) says that the states $\mathcal{O}(\mathbf{x})|0\rangle$ are mutually orthogonal for all $\mathbf{x}$. The values of $\mathbf{x}$ form a continuum, but any set of mutually orthogonal states in a separable space is countable, so the states $\mathcal{O}(\mathbf{x})|0\rangle$ cannot all belong to a single separable Hilbert space.

Intuitively, the width of a function is inversely related to the width of its Fourier transform: finer resolution in space involves larger momenta, which implies larger energies in a Lorentz-symmetric model. The fact that the correlation function (70) has zero width as a function of $\mathbf{x} - \mathbf{y}$ implies that the operator $\mathcal{O}(\mathbf{x})$ has infinitely fine resolution in space, so we should expect it to violate the energy condition. This is consistent with the result derived in section 40.

Section 17 mentioned that any operator can be made to satisfy the norm test. Example: $\mathcal{O}(\mathbf{x}) \equiv \phi(\mathbf{x})/\sqrt{\langle 0|\phi^2(\mathbf{x})|0\rangle}$ satisfies the norm test, but it also makes all the states $\mathcal{O}(\mathbf{x})|0\rangle$ orthogonal to each other in the smooth-space limit, which is inconsistent with a separable Hilbert space and with not-infinitely-fine resolution. The operator (68) is an example in which this phenomenon occurs unintentionally.

42 Correlation functions of $(e^{ic\phi})_R$

This section shows that normal-ordering fixes the problem that was described in section 41.

For each $c \in \mathbb{R}$, the normal-ordered exponential operator is[109]

$$\mathcal{O}(\mathbf{x}) \equiv (e^{ic\phi(\mathbf{x})})_R \equiv e^{ic\phi^+(\mathbf{x})} e^{ic\phi^-(\mathbf{x})}. \tag{71}$$

The operator $e^{ic\phi(\mathbf{x})}$ is unitary, but for most values of c equation (69) shows that the operator $\mathcal{O}(\mathbf{x})$ is not unitary:

$$\mathcal{O}(\mathbf{x}) = e^{ic\phi(\mathbf{x})} e^{c^2 g(\mathbf{0})/2} \qquad \Rightarrow \qquad \mathcal{O}^\dagger(\mathbf{x})\mathcal{O}(\mathbf{x}) = |e^{c^2 g(\mathbf{0})}| \neq 1. \tag{72}$$

The quantity $g(\mathbf{0})$ diverges in the smooth-space limit, so the operator $\mathcal{O}(\mathbf{x})$ fails the norm test:

$$\langle 0|\mathcal{O}^\dagger(\mathbf{x})\mathcal{O}(\mathbf{x})|0\rangle = e^{c^2 g(\mathbf{0})}. \tag{73}$$

On the other hand, its 2-point correlation function is well-behaved:[110]

$$\begin{aligned} \langle 0|\mathcal{O}^\dagger(\mathbf{x})\mathcal{O}(\mathbf{y})|0\rangle &= \langle 0|e^{-ic\phi^-(\mathbf{x})} e^{ic\phi^+(\mathbf{y})}|0\rangle \\ &= \langle 0|e^{ic\phi^+(\mathbf{y})} e^{-ic\phi^-(\mathbf{x})}|0\rangle\, e^{c^2 g(\mathbf{x}-\mathbf{y})} \qquad \text{(equation (69))} \\ &= e^{c^2 g(\mathbf{x}-\mathbf{y})}. \end{aligned} \tag{74}$$

Equation (72) says that the only difference between the normal-ordered and non-normal-ordered operators is an overall divergent numeric factor – an example of what section 18 called a *deceptive* factor.

[109] Equation (74) says that $\mathcal{O}(\mathbf{x})$ qualifies as a *renormalized* operator, as presumed by the notation in the title of this section.

[110] The fact that the smooth-space limit of this 2-point function is not a power of $|\mathbf{x}-\mathbf{y}|$ is mentioned in footnote on the text after equation (2.6) in Witten (2023).

43 $(e^{ic\phi})_R$ smeared in space: norm test

Section 42 showed that the renormalized-but-unsmeared exponential operator fails the norm test. This section shows that smearing the operator in space does not solve that problem: it still fails the norm test even when post-smeared in space.

Define $\mathcal{O}(\mathbf{x})$ as in section 42. Smear it in space to get $\mathcal{O}(f) \equiv \int_{\mathbf{x}} f(\mathbf{x})\mathcal{O}(\mathbf{x})$, and use equation (74) to get

$$\langle 0|\mathcal{O}^\dagger(f)\mathcal{O}(f)|0\rangle = \int_{\mathbf{x},\mathbf{y}} f^*(\mathbf{x})f(\mathbf{y})e^{c^2 g(\mathbf{x}-\mathbf{y})}.$$

If the smearing function is $f(\mathbf{x}) \propto e^{-\mathbf{x}^2/2\sigma^2}$, then a calculation like the one after equation (48) gives

$$\langle 0|\mathcal{O}^\dagger(f)\mathcal{O}(f)|0\rangle \propto \frac{1}{\sigma^D}\int_{\mathbf{x}} e^{-\mathbf{x}^2/4\sigma^2}e^{c^2 g(\mathbf{x})} \sim \frac{\epsilon^D}{\sigma^D} \times e^{c^2 g(\mathbf{0})}.$$

This diverges as $\epsilon \to 0$ because the exponential increases much faster than the factor ϵ^D decreases, so $\mathcal{O}(f)$ fails the norm test.

44 $(e^{ic\phi})_R$ smeared in time: average energy test

Let ϕ be a free scalar field. For each $c \in \mathbb{R}$, define the normal-ordered exponential operator

$$\mathcal{O}(\mathbf{x},t) \equiv (e^{ic\phi(\mathbf{x},t)})_R \equiv e^{ic\phi^+(\mathbf{x},t)} e^{ic\phi^-(\mathbf{x},t)}. \tag{75}$$

This generalizes (71) to any time t. Let $\mathcal{O}(f)$ denote the time-smeared operator

$$\mathcal{O}(f) \equiv \int dt\; f(t)\mathcal{O}(\mathbf{0},t). \tag{76}$$

This section shows that $\mathcal{O}(f)$ passes the average energy test.

We need to show that the ratio

$$\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} \qquad\qquad |\psi\rangle \equiv \mathcal{O}(f)|0\rangle \tag{77}$$

remains finite in the smooth-space limit. We can do this by using the expansion

$$|\psi\rangle = \int dt\; f(t) e^{ic\phi^+(\mathbf{0},t)}|0\rangle = \sum_{n\geq 0} \frac{(ic)^n}{n!}|n\rangle \tag{78}$$

with

$$|n\rangle \equiv \int dt\; f(t)\big(\phi^+(\mathbf{0},t)\big)^n|0\rangle$$

and by using the explicit expression (18) for $\phi^+(\mathbf{0},t)$. In both the numerator and denominator of (77), the only nonzero terms are those with equal numbers of energy raising and energy lowering operators. Use the smearing function $f(t) = f(0)e^{-t^2/2\sigma^2}$, the abbreviation $\omega_k \equiv \omega(\mathbf{p}_k)$, and the commutation relation for the energy raising and energy lowering operators $a^\dagger(\mathbf{p})$ and $a(\mathbf{p})$ to get

$$\begin{aligned}
\langle n|n\rangle &= n! \int_{\mathbf{p}_1,\ldots,\mathbf{p}_n} \frac{\exp\Big(-(\omega_1+\cdots+\omega_n)^2\sigma^2\Big)}{(2\omega_1)\cdots(2\omega_n)} \\
\langle n|H|n\rangle &= n! \int_{\mathbf{p}_1,\ldots,\mathbf{p}_n} \frac{\exp\Big(-(\omega_1+\cdots+\omega_n)^2\sigma^2\Big)(\omega_1+\cdots+\omega_n)}{(2\omega_1)\cdots(2\omega_n)}.
\end{aligned} \tag{79}$$

Handling the cross-terms like in section 36 and using the subscript "nct" for "no cross-terms" gives

$$\langle n|n\rangle_{\text{nct}} = n!\,\alpha^n \qquad\qquad \langle n|H|n\rangle_{\text{nct}} = n!\,n\alpha^{n-1}\beta \tag{80}$$

with

$$\alpha \equiv \int_{\mathbf{p}} \frac{e^{-\omega^2(\mathbf{p})\sigma^2}}{2\omega(\mathbf{p})} \qquad \beta \equiv \int_{\mathbf{p}} \frac{e^{-\omega^2(\mathbf{p})\sigma^2}\omega(\mathbf{p})}{2\omega(\mathbf{p})} \tag{81}$$

This gives

$$\begin{aligned}
\langle\psi|\psi\rangle_{\text{nct}} &= \sum_{n\geq 0}\left(\frac{|ic|^n}{n!}\right)^2 n!\,\alpha^n = \sum_{n\geq 0}\frac{(c^2)^n}{n!}\,\alpha^n = e^{c^2\alpha} \\
\langle\psi|H|\psi\rangle_{\text{nct}} &= \sum_{n\geq 0}\left(\frac{|ic|^n}{n!}\right)^2 n!\,n\alpha^{n-1}\beta = \beta\sum_{n\geq 1}\frac{(c^2)^n}{(n-1)!}\,\alpha^{n-1} = c^2\beta e^{c^2\alpha}.
\end{aligned}$$

Now the same reasoning about the cross-terms that was used in section 38 gives

$$\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} \leq \left.\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}\right|_{\text{nct}} = c^2\beta \propto \frac{c^2}{\sigma^D}, \tag{82}$$

which is manifestly finite in the smooth-space limit. This shows that the time-smeared normal-ordered exponential operator (76) passes the average energy test.

The result $\langle\psi|\psi\rangle \leq \langle\psi|\psi\rangle_{\text{nct}} = e^{c^2\alpha}$ with $\alpha \sim \sigma^{1-D}$ shows that the same operator also passes the norm test.

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