Conformal Isometries and the Wave Equation

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Abstract A **conformal isometry** is a transformation of spacetime that leaves the metric invariant up to an overall scale factor that may vary from one point to the next. This article shows that when $N \geq 2$, the classical wave equation in N-dimensional flat spacetime has symmetries corresponding to all conformal isometries.

Contents

1 Conformal isometries

Let $x = (x^0, x^1, ..., x^{N-1})$ denote a point in N-dimensional spacetime,[1] and write ∂_a for the partial derivative with respect to x^a. For most of this article, spacetime is taken to be flat, and the components of the metric are

$$\eta_{ab} = \begin{cases} \pm 1 & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

This includes the Minkowski metric as a special case, but the signature is arbitrary in this article. Use the abbreviations[2]

$$x \cdot y \equiv \eta_{ab}\, x^a y^a \qquad\qquad \partial \cdot \partial \equiv \eta^{ab} \partial_a \partial_b,$$

where η^{ab} are the components of the inverse metric, which in this case happen to be the same as the original components η_{ab}.

In this context, a **conformal isometry**[3] is a diffeomorphism[4] $x \to \hat{x}$ for which

$$d\hat{x} \cdot d\hat{x} = \Omega^2(x)\, dx \cdot dx. \tag{1}$$

Examples:[3]

- A conformal isometry for which $\Omega(x) = 1$ is called an **(ordinary) isometry**. Poincaré transformations are ordinary isometries in Minkowski spacetime.
- $x \to \lambda x$ with $\lambda > 0$ is called a **dilation**. In this case, $\Omega(x) = \lambda$.
- $x \to x/(x \cdot x)$ is called an **inversion**. In this case, $\Omega(x) = 1/(x \cdot x)$.

When $N \geq 3$, all other conformal isometries are generated by these three examples.[3]

[1] The superscripts are indices, not exponents.

[2] I'm using the standard summation convention, with an implied sum over each index that occurs as both a superscript and subscript in the same term.

[3] Article 38111

[4] In this context, a **diffeomorphism** is a smooth rearrangement of the points of spacetime (article 93875), but not necessarily defined at all points in spacetime. The inversion $x \to x/(x \cdot x)$ is defined only where $x \cdot x \neq 0$.

2 Symmetries of the wave equation

This article is about symmetries of the **wave equation**

$$\partial\cdot\partial\phi(x)=0 \tag{2}$$

in N-dimensional spacetime, where $\phi(x)$ is a classical scalar field. The goal is to show that if $\phi(x)$ satisfies the wave equation with $N\geq 2$ and if $x\to\hat{x}$ is any conformal isometry (equation (1)), then the field $\hat{\phi}(x)$ defined by[5]

$$\hat{\phi}(x)\equiv\omega(x)\phi\big(\hat{x}(x)\big) \qquad\qquad \omega(x)\equiv\big(\Omega(x)\big)^{(N-2)/2} \tag{3}$$

also satisfies the wave equation at all points where $\hat{x}$ is defined:

$$\partial\cdot\partial\hat{\phi}(x)=0. \tag{4}$$

The transformation $\phi(x)\to\hat{\phi}(x)$ defined by equation (3) is sometimes called a **conformal transformation**, but that name is also used for other things.[6] To avoid equivocation, this article will call it a **conformal fieldomorphism**.[7,8] The goal is to show that if $N\geq 2$, then all conformal fieldomorphisms are symmetries of the wave equation.[9]

[5] Equation (1) doesn't specify the sign of $\Omega(x)$, but equation (3) assumes the convention $\Omega(x)>0$.

[6] Appendix D in Wald (1984) uses the name *conformal transformation* for what many physicists (including this article) call a *Weyl transformation*. Section 7.6.2 in Nakahara (1990) uses the name *conformal transformation* for what this article calls a *conformal isometry*.

[7] This name is not standard. It builds on the name **fieldomorphism** that article 00418 used for a transformation like $\phi(x)\to\hat{\phi}(x)\equiv\phi(\hat{x})$, where $x\to\hat{x}$ is an arbitrary diffeomorphism.

[8] Section 15 reviews the concept of a *Weyl transformation*, mainly for the purpose of distinguishing it from what this article calls a conformal fieldomorphism.

[9] When $N\geq 3$, they are also symmetries of the more general equation $\partial\cdot\partial\phi\propto\phi^{(N+2)/(N-2)}$. Notice that the exponent is an integer only if $N=4$ or $N=6$.

3 Symmetries of the wave equation: examples

Article 49705 showed that if $x \to \hat{x}$ is any diffeomorphism for which $d\hat{x}\cdot d\hat{x} = dx\cdot dx$, then the corresponding fieldomorphism (3) is a symmetry of the wave equation. In this case, the factor $\omega(x)$ is equal to 1.

The easiest example with $\omega(x) \neq 1$ is the conformal fieldomorphism corresponding to a dilation $x \to \lambda x$ with constant scale factor $\lambda \neq 1$. The fact that this is a symmetry of the wave equation should be clear by inspection. The factor $\omega(x)$ is still independent of x in this case.

The easiest example in which $\omega(x)$ is not independent of x is the conformal fieldomorphism correponding to an inversion $x \to x/(x \cdot x)$. In this case, the scale function in equation (1) turns out to be[10] $\Omega(x) = 1/(x\cdot x)$, so the transformed field is

$$\hat{\phi}(x) = \omega(x)\,\phi\left(\frac{x}{x\cdot x}\right) \qquad\qquad \text{with } \omega(x) = \frac{1}{(x\cdot x)^{(N-2)/2}}. \tag{5}$$

This is defined wherever $x \cdot x \neq 0$. Straightforward calculation shows that this function $\omega(x)$ satisfies

$$\begin{gathered} \partial\cdot\partial\omega(x) = 0 \\ \partial\cdot\partial\big(\omega(x)\hat{x}\big) = 0 \end{gathered}$$

wherever $x \cdot x \neq 0$. Section 6 will use these properties of $\omega(x)$ to show that the transformation $\phi(x) \to \hat{\phi}(x)$ defined by (5) is a symmetry of the wave equation.

[10] Article 38111

4 Composing fieldomorphisms

The composition of two conformal isometries (the result of applying them sequentially) is another conformal isometry. This is clear from the definition (1). This section shows that the composition of two conformal fieldomorphisms (3) is another conformal fieldomorphism that agrees with the composition of the corresponding conformal isometries.

Since we're dealing with two conformal isometries now, we need to use a notation that distinguishes between them. If we use the operator-like notation

$$x \to \sigma_1 x \qquad\qquad x \to \sigma_2 x$$

for the two conformal isometries, then their composition (the result of applying σ_1 and then σ_2) is

$$x \to \sigma_{12} x \equiv \sigma_2 \sigma_1 x.$$

These are conformal isometries, so the effect of each one on the line element may be written

$$\begin{aligned} d(\sigma_1 x) \cdot d(\sigma_1 x) &= \Omega_1^2(x)\, dx \cdot dx \\ d(\sigma_2 x) \cdot d(\sigma_2 x) &= \Omega_2^2(x)\, dx \cdot dx \\ d(\sigma_2 \sigma_1 x) \cdot d(\sigma_2 \sigma_1 x) &= \Omega_{12}^2(x)\, dx \cdot dx. \end{aligned}$$

Use the first two equations to get

$$\begin{aligned} d(\sigma_2 \sigma_1 x) \cdot d(\sigma_2 \sigma_1 x) &= \Omega_2^2(\sigma_1 x)\, d(\sigma_1 x) \cdot d(\sigma_1 x) \\ &= \Omega_2^2(\sigma_1 x) \Omega_1^2(x)\, dx \cdot dx, \end{aligned}$$

which gives this equation for the scale function Ω_{12} of the composite transformation:

$$\Omega_{12}(x) = \Omega_2(\sigma_1 x) \Omega_1(x). \tag{6}$$

Now, for any function $\phi(x)$ and any $n \in \{1, 2, 12\}$, define a transformation $\phi \to \sigma_n \phi$ by

$$\sigma_n \phi(x) \equiv \omega_n(x) \phi(\sigma_n x) \tag{7}$$

with

$$\omega_n(x) \equiv \big(\Omega_n(x)\big)^{(N-2)/2}, \tag{8}$$

as in equation (3). Equations (7) with $n \in \{1,2\}$ imply

$$\sigma_2\sigma_1\phi(x) = \omega_1(x)\sigma_2\phi(\sigma_1 x) = \omega_1(x)\omega_2(\sigma_1 x)\phi(\sigma_2\sigma_1 x),$$

and comparing this to equation (7) with $n = 12$ gives

$$\omega_{12}(x) = \omega_2(\sigma_1 x)\omega_1(x), \tag{9}$$

which is consistent with equations (6) and (8). This completes the derivation.

This would all still be true if $\omega_n(x) = \big(\Omega_n(x)\big)^E$ for any exponent E, but the exponent shown in (8) is special because it makes $\omega_n(x)$ satisfy the conditions (10) and (11). This is easy to check by direct calculation when σ_n is the inversion $\sigma_n x = x/(x \cdot x)$. The following sections explain how it can be inferred for other conformal isometries.

The result derived in this section can be expressed using the language of category theory.[11] Conformal isometries may be regarded as the morphisms in a category with just one object. That one object is the smooth manifold (spacetime) on which the conformal isometries act. Conformal fieldomorphisms may be regarded as the morphisms in another category with just one object. That one object is the set of scalar fields that satisfy the wave equation (wherever they're defined). In both cases, the composition of two morphisms is another morphism, as required by the definition of **category**. By expressing conformal fieldomoprhisms in terms of conformal isometries, equations (3) define a **functor** from the conformal-isometries category to the conformal-fieldomorphisms category. A functor converts morphisms in one category to morphisms in another category, respecting composition.

[11] Spivak (2013) and McLarty (1992) are relatively inviting introductions to category theory.

5 Strategy

The goal is to show that the transformations $\phi \to \hat{\phi}$ defined in equation (3) – conformal fieldomorphisms – are symmetries of the wave equation (2) if $N \geq 2$. Here's an outline:

- Section 6 will show that a conformal fieldomorphism is a symmetry of the wave equation if ω satisfies the conditions

$$\partial \cdot \partial \omega(x) = 0 \tag{10}$$

$$\partial \cdot \partial \big(\omega(x)\hat{x}\big) = 0 \tag{11}$$

  wherever it is defined.

- Sections 7-8 will use a different method, involving the action principle, to show that a conformal fieldomorphism is a symmetry of the wave equation if ω satisfies the condition (10).[12]

- Sections 9-10 will show that if two conformal isometries both satisfy the conditions (10)-(11), then so does their composition.[13]

- Sections 11 use that result to show that every conformal isometry satisfies the conditions (10)-(11) if $N \geq 2$. This implies that all of the corresponding conformal fieldomorphisms are symmetries of the wave equation.

For extra fun, sections 12-14 review how the embedding space formalism can be used to study conformal fieldomorphisms.[14]

[12] This approach doesn't use the condition (11) explicitly, but it does use the fact that $\omega(x)$ is defined by equations (1) and (3), which the subsequent sections use to show that ω satisfies the conditions (10) and (11).

[13] Actually the proof shown here is incomplete, because the composition of two conformal isometries may be defined everywhere (by continuation) even if the conformal isometries that went into it are not. Example: composing the inversion $x \to x/(x \cdot x)$ with itself gives the identity transformation, which is defined everywhere, in the same sense that $(x \cdot x)/(x \cdot x) = 1$. The proof shown here doesn't include the continuation step.

[14] Article 38111 uses the embedding space formalism to study conformal isometries (equation (1)).

6 Direct approach

This section shows that if $\phi(x)$ is any solution of the wave equation (2), and if ω satisfies the conditions (10) and (11),[15] then the new field $\hat{\phi}(x)$ defined in (3) is another solution of the wave equation.

Start with the elementary identities

$$\begin{aligned}\partial\cdot\partial\hat{\phi}(x) &= \partial\cdot\partial\big(\omega(x)\phi(\hat{x})\big)\\ &= \big(\partial\cdot\partial\omega(x)\big)\phi(\hat{x}) + 2\big(\partial\omega(x)\big)\cdot\partial\phi(\hat{x}) + \omega(x)\partial\cdot\partial\phi(\hat{x}).\end{aligned} \tag{12}$$

Use the abbreviation $\hat{\partial}_a \equiv \partial/\partial\hat{x}^a$ to get

$$\partial\phi(\hat{x}) = (\partial\hat{x}^a)\hat{\partial}_a\phi(\hat{x}) \tag{13}$$

$$\begin{aligned}\partial\cdot\partial\phi(\hat{x}) &= \partial\cdot\big((\partial\hat{x}^a)\hat{\partial}_a\phi(\hat{x})\big)\\ &= (\partial\cdot\partial\hat{x}^a)\hat{\partial}_a\phi(\hat{x}) + (\partial\hat{x}^a)\cdot\partial\hat{\partial}_a\phi(\hat{x})\\ &= (\partial\cdot\partial\hat{x}^a)\hat{\partial}_a\phi(\hat{x}) + (\partial\hat{x}^a)\cdot(\partial\hat{x}^c)\hat{\partial}_c\hat{\partial}_a\phi(\hat{x})\end{aligned} \tag{14}$$

To continue, use the general identity

$$d\hat{x}^a = dx^b\,(\partial_b\hat{x}^a) \tag{15}$$

to see that (1) implies

$$\eta_{cd}(\partial_a\hat{x}^c)(\partial_b\hat{x}^d) = \Omega^2(x)\eta_{ab}, \tag{16}$$

which in turn implies[16]

$$(\partial\hat{x}^a)\cdot(\partial\hat{x}^b) = \Omega^2(x)\eta^{ab}. \tag{17}$$

Equation (17) implies

$$(\partial\hat{x}^a)\cdot(\partial\hat{x}^c)\hat{\partial}_c\hat{\partial}_a = \Omega^2(x)\hat{\partial}\cdot\hat{\partial},$$

[15] For the rest of this article, the qualification "wherever the conformal isometry is defined" is understood.

[16] To derive this, let η be the matrix with components η_{ab}, and let M be the matrix with components $M_{ab} \equiv \partial_a\hat{x}^b$. Then equation (16) is $M\eta M^T = \Omega^2\eta$. Take the matrix inverse of both sides and then re-arrange to get $M^T\eta^{-1}M = \Omega^2\eta^{-1}$, which is equation (17).

and the assumption that $\phi(x)$ satisfies the wave equation ($\partial\cdot\partial\phi(x) = 0$) implies $\hat{\partial}\cdot\hat{\partial}\phi(\hat{x}) = 0$ just by relabeling the coordinates, so the last term in equation (14) is zero. Use these results in (12) to get

$$\partial\cdot\partial\hat{\phi}(x) = \big(\partial\cdot\partial\omega(x)\big)\phi(\hat{x}) + \Gamma^a\hat{\partial}_a\phi(\hat{x}) \tag{18}$$

with

$$\Gamma^a \equiv 2\big(\partial\omega(x)\big)\cdot(\partial\hat{x}^a) + \omega(x)(\partial\cdot\partial\hat{x}^a). \tag{19}$$

The condition (10) implies that the quantity Γ^a may also be written

$$\Gamma^a = \partial\cdot\partial\big(\omega(x)\hat{x}^a\big). \tag{20}$$

Equations (18) and (20) show that $\hat{\phi}(x)$ satisfies the wave equation if $\omega(x)$ satisfies the conditions (10) and (11).

7 Approach using the action principle

This section shows that if $\phi(x)$ is any solution of the wave equation (2), and if ω satisfies the condition (10), then the new field $\hat{\phi}(x)$ defined in (3) is another solution of the wave equation. In contrast to the approach that was used in section 6, the approach used here involves only first-order derivatives of the field.

The wave equation (2) can be written as[17]

$$\frac{\delta S[\phi]}{\delta\phi(x)} = 0$$

where $S[\phi]$ is the **action**[18]

$$S[\phi] = \int d^N x \; (\partial\phi)\cdot(\partial\phi). \tag{21}$$

Now suppose that $x \to \hat{x}$ is any conformal isometry, not necessarily satisfying the condition (10). Section 8 shows that replacing the original scalar field $\phi(x)$ with the new scalar field (3) has this effect on the action:

$$S[\hat{\phi}] = \pm S[\phi] + \int d^N x \; \partial\cdot(\text{something}) - \int d^N x \; \phi^2(\hat{x})\omega(x)\partial\cdot\partial\omega(x). \tag{22}$$

Adding a total-derivative term to S doesn't affect the equation of motion (21), so the result (22) implies that if $\phi(x)$ satisfies the wave equation and ω satisfies the condition (10), then $\hat{\phi}(x)$ also satisfies the wave equation.

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[17] Article 49705

[18] I'm omitting a conventional but inconsequential overall factor of 1/2.

8 Approach using the action principle: details

To derive (22), start with

$$S[\hat{\phi}] = \int d^N x\ \big(\partial\hat{\phi}(x)\big)\cdot\big(\partial\hat{\phi}(x)\big). \tag{23}$$

Equation (3) gives

$$\partial_a\hat{\phi}(x) = \omega(x)\partial_a\phi(\hat{x}) + \phi(\hat{x})\partial_a\omega(x),$$

and using this in (23) gives

$$S[\hat{\phi}] = S_1[\phi] + S_2[\phi] \tag{24}$$

with

$$S_1[\phi] \equiv \int d^N x\ \omega^2(x)\big(\partial\phi(\hat{x})\big)\cdot\big(\partial\phi(\hat{x})\big) \tag{25}$$

and

$$\begin{aligned} S_2[\phi] &\equiv \int d^N x\ \Big[2\omega(x)\phi(\hat{x})\big(\partial\phi(\hat{x})\big)\cdot\big(\partial\omega(x)\big) + \phi^2(\hat{x})\big(\partial\omega(x)\big)\cdot\big(\partial\omega(x)\big)\Big] \\ &= \int d^N x\ \partial\cdot\big(\omega(x)\phi^2(\hat{x})\partial\omega(x)\big) - \int d^N x\ \phi^2(\hat{x})\omega(x)\partial\cdot\partial\omega(x). \end{aligned} \tag{26}$$

To get a more useful expression for $S_1[\phi]$, use the general identity

$$\partial_a = (\partial_a\hat{x}^c)\hat{\partial}_c$$

to write it as

$$S_1[\phi] = \int d^N x\ \omega^2(x)(\partial\hat{x}^a)\cdot(\partial\hat{x}^b)\big(\hat{\partial}_a\phi(\hat{x})\big)\big(\hat{\partial}_b\phi(\hat{x})\big). \tag{27}$$

Use (17) in (27) to get

$$S_1[\phi] = \int d^N x\ \omega^2(x)\Omega^2(x)\big(\hat{\partial}\phi(\hat{x})\big)\cdot\big(\hat{\partial}\phi(\hat{x})\big). \tag{28}$$

The identity (15) implies

$$d^N x = \frac{d^N \hat{x}}{|\partial \hat{x}|} \tag{29}$$

where $|\partial \hat{x}|$ is the determinant of the matrix with components $\partial_b \hat{x}^a$, and equation (16) implies

$$|\partial \hat{x}|^2 = \big(\Omega^2(x)\big)^N. \tag{30}$$

Use (29), (30), and the definition of ω in (28) to get

$$S_1[\phi] = \pm \int d^N \hat{x}\ \big(\hat{\partial}\phi(\hat{x})\big) \cdot \big(\hat{\partial}\phi(\hat{x})\big) = \pm S[\phi]. \tag{31}$$

Combining (24), (26), and (31) gives the promised result (22).

9 Condition (10)

As in section 4, consider two conformal isometries σ_1 and σ_2. This section shows that if $\omega_1(x)$ and $\omega_2(x)$ both satisfy condition (10), then so does $\omega_{12}(x)$. The notation here is the same as in section 4, and the abbreviations

$$\hat{x} \equiv \sigma_1 x \qquad\qquad \hat{\partial} \equiv \frac{\partial}{\partial \hat{x}}$$

will also be used.

Equation (9) combined with $\partial \cdot \partial \omega_1(x) = 0$ implies

$$\begin{aligned}
\partial \cdot \partial \omega_{12}(x) &= 2\big(\partial \omega_1(x)\big) \cdot \partial \omega_2(\sigma_1 x) + \omega_1(x) \partial \cdot \partial \omega_2(\sigma_1 x) \\
&= 2\big(\partial \omega_1(x)\big) \cdot (\partial \hat{x}^a) \hat{\partial}_a \omega_2(\hat{x}) + \omega_1(x) \partial \cdot \big((\partial \hat{x}^a) \hat{\partial}_a \omega_2(\hat{x})\big) \\
&= 2\big(\partial \omega_1(x)\big) \cdot (\partial \hat{x}^a) \hat{\partial}_a \omega_2(\hat{x}) + \omega_1(x) (\partial \cdot \partial \hat{x}^a) \hat{\partial}_a \omega_2(\hat{x}) \\
&\qquad\qquad\qquad + \omega_1(x) (\partial \hat{x}^b) \cdot (\partial \hat{x}^a) \hat{\partial}_b \hat{\partial}_a \omega_2(\hat{x}).
\end{aligned}$$

Use equation (17) and $\hat{\partial} \cdot \hat{\partial} \omega_2(\hat{x}) = 0$ to see that the last term is zero, which leaves

$$\begin{aligned}
\partial \cdot \partial \omega_{12}(x) &= \big[2\big(\partial \omega_1(x)\big) \cdot (\partial \hat{x}^a) + \omega_1(x)(\partial \cdot \partial \hat{x}^a)\big]\, \hat{\partial}_a \omega_2(\hat{x}) \\
&= \big[\partial \cdot \partial\big(\omega_1(x) \hat{x}^a\big) - \hat{x}^a \partial \cdot \partial \omega_1(x))\big]\, \hat{\partial}_a \omega_2(\hat{x}).
\end{aligned}$$

The assumption that $\omega_1(x)$ satisfies the conditions (10) and (11) implies that the quantity in square brackets is zero, so this proves that $\omega_{12}(x)$ satisfies condition (10).

10 Condition (11)

This section shows that if $\omega_1(x)$ and $\omega_2(x)$ both satisfy condition (11), then so does $\omega_{12}(x)$. More explicitly: if

$$\partial\cdot\partial\big(\omega_1(x)\sigma_1 x\big) = 0 \tag{32}$$

$$\partial\cdot\partial\big(\omega_2(x)\sigma_2 x\big) = 0, \tag{33}$$

then

$$\partial\cdot\partial\big(\omega_{12}(x)\sigma_{12} x\big) = 0. \tag{34}$$

The notation here is the same as in section 9.

Equation (9) combined with $\partial\cdot\partial\omega_1(x) = 0$ implies

$$\begin{aligned}\partial\cdot\partial\big(\omega_{12}(x)(\sigma_{12}x)^a\big) &= \partial\cdot\Big[\omega_2(\hat{x})(\sigma_2\hat{x})^a\partial\omega_1(x) + \omega_1(x)\partial\big(\omega_2(\hat{x})(\sigma_2\hat{x})^a\big)\Big] \\ &= \partial\cdot\Big[\omega_2(\hat{x})(\sigma_2\hat{x})^a\partial\omega_1(x) + \omega_1(x)(\partial\hat{x}^b)\hat{\partial}_b\big(\omega_2(\hat{x})(\sigma_2\hat{x})^a\big)\Big]\end{aligned}$$

Use equations (17) and (33) in the last term to get

$$\partial\cdot\partial\big(\omega_{12}(x)(\sigma_{12}x)^a\big) = \partial\cdot\left[\omega_2(\hat{x})(\sigma_2\hat{x})^a\partial\omega_1(x)\right] + \partial\cdot\Big[\omega_1(x)(\partial\hat{x}^b)\Big]\,\hat{\partial}_b\big(\omega_2(\hat{x})(\sigma_2\hat{x})^a\big).$$

Use $\partial\cdot\partial\omega_1(x) = 0$ in the first term to get

$$\begin{aligned}\partial\cdot\partial\big(\omega_{12}(x)(\sigma_{12}x)^a\big) &= \partial\left[\omega_2(\hat{x})(\sigma_2\hat{x})^a\right]\cdot\partial\omega_1(x) + \partial\cdot\Big[\omega_1(x)(\partial\hat{x}^b)\Big]\,\hat{\partial}_b\big(\omega_2(\hat{x})(\sigma_2\hat{x})^a\big) \\ &= (\partial\hat{x}^b)\hat{\partial}_b\left[\omega_2(\hat{x})(\sigma_2\hat{x})^a\right]\cdot\partial\omega_1(x) + \partial\cdot\Big[\omega_1(x)(\partial\hat{x}^b)\Big]\,\hat{\partial}_b\big(\omega_2(\hat{x})(\sigma_2\hat{x})^a\big),\end{aligned}$$

and use $\partial\cdot\partial\omega_1(x) = 0$ again to get

$$\partial\cdot\partial\big(\omega_{12}(x)(\sigma_{12}x)^a\big) = \hat{\partial}_b\big(\omega_2(\hat{x})(\sigma_2\hat{x})^a\big)\,\Big[\partial\cdot\partial\big(\omega_1(x)\hat{x}^b\big)\Big]\,.$$

Equation (32) implies that the quantity in square brackets is zero, which proves (34).

11 The main result

When $N \geq 3$, the group of conformal isometries is generated by ordinary isometries, dilations, and inversions.[19] We already know[20] that these transformations satisfy the conditions (10) and (11), so the results derived in sections 6-10 imply that all conformal isometries correspond – via (3) – to symmetries of the wave equation when $N \geq 3$.

When $N = 2$, ω is independent of x, so the conditions (10) and (11) reduce to the condition

$$\partial \cdot \partial \hat{x} = 0. \tag{35}$$

The line element may be written

$$dx \cdot dx \propto du\, dv$$

with

$$u \equiv x^0 + x^1 \qquad\qquad v \equiv x^0 - x^1$$

if the signature is Lorentzian (Minkowski spacetime), or with

$$u \equiv x^0 + ix^1 \qquad\qquad v \equiv x^0 - ix^1$$

if the signature is Euclidean, where $i^2 = -1$. In either case, the condition (35) may be written[21]

$$\frac{\partial}{\partial u}\frac{\partial}{\partial v}\,\hat{u} = 0 \qquad\qquad \frac{\partial}{\partial u}\frac{\partial}{\partial v}\,\hat{v} = 0.$$

According to sections 2.3 and 2.5 Schottenloher (2008), all conformal isometries satisfy these conditions when $N = 2$, so the conclusion of the preceding paragraph extends to $N \geq 2$.

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[19] Section 1

[20] Section 3

[21] In the Euclidean case, $\partial/\partial u \equiv \frac{1}{2}(\partial_0 - i\partial_1)$ and $\partial/\partial v \equiv \frac{1}{2}(\partial_0 + i\partial_1)$.

12 A quick review of the embedding space formalism

Article 38111 introduces the **embedding space formalism**, which relates at least some conformal isometries of the original N-dimensional spacetime $\mathcal{M}$ to origin-preserving ordinary isometries of an $(N+2)$-dimensional spacetime in which $\mathcal{M}$ is embedded.[22] This section reviews the idea, and section 13 applies it to the wave equation.

Consider an $(N+2)$-dimensional manifold with coordinates $X \equiv (X^0, X^1, ..., X^{N-1})$ and Y and Z, where Y and Z are individual real-valued coordinates, and with the metric defined implicitly by the line element[23]

$$dX \cdot dX + dY^2 - dZ^2. \tag{36}$$

I'll call this the **ambient** manifold.[24,25] Let $\mathcal{C}$ be the "cone" defined by

$$X \cdot X + Y^2 - Z^2 = 0, \tag{37}$$

and let $\mathcal{P}$ be the hyperplane defined by

$$Y + Z = R \tag{38}$$

for some constant R. We can use

$$x^a \equiv \frac{X^a}{Y+Z} \tag{39}$$

as coordinates on the N-dimensional intersection $\mathcal{C} \cap \mathcal{P}$, everywhere except where $Y + Z = 0$. Article 38111 shows that the induced metric in this N-dimensional manifold is conformally equivalent the Minkowski metric $dx \cdot dx$ and that origin-preserving ordinary isometries of the ambient manifold correspond to conformal isometries of the N-dimensional manifold.

[22] Weinberg (2010) describes how tensor fields in Minkowski spacetime can be represented using this formalism.

[23] As in the preceding sections, $X \cdot X \equiv \eta_{ab} X^a X^b$ for any quantity X with N components.

[24] It's also called the **embedding space**, because the spacetime of interest – Minkowski spacetime – will be embedded inside it.

[25] Mnemonic: I'm using uppercase letters for coordinates in the bigger (higher-dimensional) manifold, and lowercase letters for coordinates in the smaller manifold (Minkowski spacetime).

13 Embedding space and the wave equation

The embedding space formalism can be used to relate the symmetries of the wave equation in N-dimensional spacetime to ordinary isometries in the $(N+2)$-dimensional embedding space. Let $\Phi(X,Y,Z)$ be a function that satisfies this $(N+2)$-dimensional version of the wave equation:

$$\partial_X\cdot\partial_X\Phi+\partial_Y^2\Phi-\partial_Z^2\Phi=0. \tag{40}$$

All origin-preserving isometries of (36) correspond to symmetries of equation (40) in the usual way. To relate these to the symmetries (3) of the wave equation in N-dimensional Minkowski spacetime, consider functions of the form[26]

$$\Phi(X,Y,Z)=(Y+Z)^{-(N-2)/2}\,f\left(\frac{X}{Y+Z},\,\frac{Y-Z}{Y+Z}\right). \tag{41}$$

Section 14 shows that applying the differential operator

$$\partial_X\cdot\partial_X+\partial_Y^2-\partial_Z^2 \tag{42}$$

to such a function and then imposing the constraints (37) and (38) gives the same result as applying the differential operator $\partial_x\cdot\partial_x$ to

$$\phi(x)\equiv R^{-(N+2)/2}\,f\left(x,\,\frac{-x\cdot x}{R^2}\right). \tag{43}$$

In particular, if the function (41) satisfies (40) everywhere on $\mathcal{C}\cap\mathcal{P}$, then the function (43) satisfies the usual wave equation in Minkowski spacetime.

[26] This is equation (37) in Bars (2000), but beware: that paper uses the same symbol λ for two different things on the same page. The two different meanings of λ are introduced implicitly in equations (26) and (31) of that paper.

14 Embedding space and the wave equation: details

This section shows that applying the differential operator (42) to any function of the form (41) and then imposing the constraints (37) and (38) gives the same result as applying the differential operator $\partial_x \cdot \partial_x$ to the function (43).[27]

The coordinates X, Y, Z can be written as

$$Y + Z = \sigma \qquad\qquad Y - Z = \rho \qquad\qquad X = \sigma x$$

wherever $\sigma \neq 0$. This is consistent with (39). Use the generic identity[28]

$$\sum_a dX^a \frac{\partial}{\partial X^a} + dY\,\partial_Y + dZ\,\partial_Z = \sum_a dx^a\,\frac{\partial}{\partial x^a} + d\rho\,\partial_\rho + d\sigma\,\partial_\sigma \tag{44}$$

to infer[29]

$$\begin{aligned}
\frac{\partial}{\partial x^a} &= \sigma \frac{\partial}{\partial X^a} \\
\frac{\partial}{\partial \rho} &= \frac{1}{2}(\partial_Y - \partial_Z) \\
\frac{\partial}{\partial \sigma} &= \sum_a x^a \frac{\partial}{\partial X^a} + \frac{1}{2}(\partial_Y + \partial_Z) = \frac{1}{\sigma}\sum_a x^a \frac{\partial}{\partial x^a} + \frac{1}{2}(\partial_Y + \partial_Z)
\end{aligned}$$

and uses these to infer that the differential operator in (40) may be written

$$\begin{aligned}
\partial_X \cdot \partial_X + \partial_Y^2 - \partial_Z^2 &= \partial_X \cdot \partial_X + (\partial_Y + \partial_Z)(\partial_Y - \partial_Z) \\
&= \frac{1}{\sigma^2}\partial_x \cdot \partial_x + 4\left(\frac{\partial}{\partial \sigma} - \frac{1}{\sigma}\sum_a x^a \frac{\partial}{\partial x^a}\right)\frac{\partial}{\partial \rho}.
\end{aligned}$$

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[27] The approach used here is a slight variation of the approach used in Bars (2000), section 4.1.

[28] This identity holds for any two coordinate systems X, Y, Z and x, ρ, σ, no matter how they're related to each other (as long as each one can be written in terms of the other).

[29] To deduce this, write dX, dY, dZ in terms of $dx, d\rho, d\sigma$ and substitute those expressions for dX, dY, dZ into the left-hand side of (44).

Equation (41) may also be written

$$\Phi = \sigma^{-(N-2)/2} f(x, \rho/\sigma).$$

Any function of this form satisfies

$$\begin{aligned}\left(\frac{\partial}{\partial\sigma} - \frac{1}{\sigma}\sum_a x^a \frac{\partial}{\partial x^a}\right)\frac{\partial}{\partial\rho}\Phi &= \left(\frac{\partial}{\partial\sigma} - \frac{1}{\sigma}\sum_a x^a \frac{\partial}{\partial x^a}\right)\frac{1}{\sigma}\Phi_1 \\ &= \frac{-1}{\sigma^2}\left(\frac{N}{2}\Phi_1 + \frac{\rho}{\sigma}\Phi_2 + \sum_a x^a \frac{\partial}{\partial x^a}\Phi_1\right) \end{aligned} \tag{45}$$

with

$$\begin{aligned}\Phi_1 &\equiv \sigma^{-(N-2)/2}\,\frac{\partial}{\partial y} f(x, y)\bigg|_{y=\rho/\sigma} \\ \Phi_2 &\equiv \sigma^{-(N-2)/2}\,\frac{\partial^2}{\partial y^2} f(x, y)\bigg|_{y=\rho/\sigma}.\end{aligned}$$

This is true even if Φ doesn't satisfy (40). On the other hand, any function of the form (43) satisfies

$$\partial_x \cdot \partial_x f(x, y) = \left[\partial_x \cdot \partial_x f(x, y) - \frac{N}{2} f_1(x, y) + \frac{x \cdot x}{R^2} f_2(x, y) - \sum_a x^a \frac{\partial}{\partial x^a} f_1(x, y)\right]_{y=-x\cdot x/R^2}$$

with

$$f_1(x, y) \equiv \frac{\partial}{\partial y} f(x, y) \qquad\qquad f_2(x, y) \equiv \frac{\partial^2}{\partial y^2} f(x, y).$$

According to equations (37) and (38), points on the intersection $\mathcal{C} \cap \mathcal{P}$ satisfy

$$\sigma = R \qquad\qquad \rho = \frac{-x \cdot x}{R},$$

Use this in the last line of equation (45) to complete the derivation.

15 Weyl invariance

Section 2 defined some symmetries of the wave equation. Those symmetries mix the scalar field's values at different points of spacetime with each other in the same way that conformal isometries mix the points of spacetime itself, and they don't transform the metric field at all.

In contrast, a **Weyl transformation**[30] does affect the metric field but doesn't mix the fields' values in different regions with each other:

$$\phi(x) \to \Omega^s(x)\phi(x) \qquad\qquad g_{ab}(x) \to \Omega^2(x)g_{ab}(x) \tag{46}$$

where $s \in \mathbb{R}$ is called the **conformal weight** of the scalar field. To say anything about what a Weyl transformation does to the action (or equation of motion) for a scalar field, we first need to specify how the action depends on the metric field. One natural choice is[31]

$$S[g,\phi] = \int d^N x\ \sqrt{|g|}\, g^{ab}(\partial_a\phi)(\partial_b\phi) \tag{47}$$

where $g_{ab}(x)$ are the components of a metric tensor, $g^{ab}(x)$ are the components of its inverse, and $|g|(x)$ is its determinant. When $N = 2$, the action (47) is invariant under Weyl transformations (46) with $s = 0$.[32] When $N \neq 2$, the action (47) is not invariant under Weyl transformations (46) for any s, but the modified action

$$S'[g,\phi] = \int d^N x\ \sqrt{|g|}\ \left(g^{ab}(\partial_a\phi)(\partial_b\phi) + \frac{N-2}{4(N-1)}R\phi^2\right) \tag{48}$$

is invariant up to a total derivative under Weyl transformations with[33,34] $s =$

[30] Article 38111

[31] Section 16 reviews why this choice is natural.

[32] In more detail: the only part of the integrand affected by this transformation is the metric-dependent factor $\sqrt{|g|}\, g^{ab}$, and the transformation (46) leaves this invariant when $N = 2$.

[33] Section 22.3 in Blau (2021). Beware that Blau quietly ignores the total-derivative term, acknowledging its existence only in the text above equation (22.115).

[34] Appendix D in Wald (1984) derives this using the equation of motion instead of the action. The equation of motion is not affected by the total-derivative term that a typical Weyl transformation adds to the action.

$(2-N)/2$ when $N \geq 2$, where R is the Ricci scalar constructed from the curvature tensor.[35] This property of (48) is called **Weyl invariance** or **conformal invariance**,[36] but beware that the names *conformal transformation*, *conformal invariance*, and *conformal symmetry* are all overloaded in the physics literature.

A symmetry that doesn't mix the fields' values in different regions with each other is called an **internal symmetry**, so Weyl invariance is an example of an internal symmetry.

[35] This is not related to the constant R in section 12.

[36] Wald (1984), appendix D

16 General covariance

The action (47) is a natural choice because it has a property that is sometimes called **general covariance**: for any diffeomorphism $x \to \hat{x}$, replacing the original fields $\phi(x)$ and $g_{ab}(x)$ with the new fields

$$\hat{\phi}(x) \equiv \phi(\hat{x}) \qquad\qquad \hat{g}_{ab}(x) \equiv g_{cd}(\hat{x})(\partial_a \hat{x}^c)(\partial_b \hat{x}^d) \tag{49}$$

leaves the action invariant:

$$S[\hat{g}, \hat{\phi}] = S[g, \phi].$$

To derive this, use the identities[37]

$$\partial_a \hat{\phi}(x) = (\partial_a \hat{x}^b)\hat{\partial}_b \phi(\hat{x}) \qquad\qquad \hat{g}^{ab}(x) = g^{cd}(\hat{x})(\hat{\partial}_c x^a)(\hat{\partial}_d x^b) \tag{50}$$

to get

$$\hat{g}^{ab}(x)\big(\partial_a \hat{\phi}(x)\big)\big(\partial_b \hat{\phi}(x)\big) = g^{ab}(\hat{x})\big(\hat{\partial}_a \phi(\hat{x})\big)\big(\hat{\partial}_b \phi(\hat{x})\big),$$

and use the definition of $\hat{g}_{ab}(x)$ to get

$$d^N x\, \sqrt{|\hat{g}(x)|} = d^N \hat{x}\, \sqrt{|g(\hat{x})|}.$$

The modified action (47) has this property, too.

[37] To derive the second identity, let $G(\hat{x})$ and $\hat{G}(x)$ be the matrices with components $g_{ab}(\hat{x})$ and $\hat{g}_{ab}(x)$, respectively. Let M be the matrix with components $M_{ab} = \partial_a \hat{x}^b$. Then the second equation in (49) is $\hat{G}(x) \equiv MG(\hat{x})M^T$. Take the matrix inverse of both sides to get $\hat{G}^{-1}(x) = (M^T)^{-1}G^{-1}(\hat{x})M^{-1}$. This is the second equation in (50).

17 References

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