cphysics.org article **10142** 2024-02-25

Universality and Continuum Limits with Scalar Quantum Fields

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Abstract In quantum field theory, a model's predictions at sufficiently low resolution often depend only on relatively few details of the model's construction. This is called **universality**. Thanks to universality, finding models that accurately reproduce the results of low-resolution measurements doesn't require getting all of the high-resolution details exactly right. The key is knowing which combinations of high-resolution details do matter at much lower resolutions. This article introduces some of the basic ideas that lead to that insight, emphasizing how those ideas relate to the act of taking (near-)continuum limits of models that were initially defined by treating spacetime as a lattice. Models that involve only scalar fields are used as examples.

Contents

1 Introduction

Real measurements have limited resolution. To achieve finer resolution, we need to use more energy.[1] This limitation of real measurements is important in quantum field theory (QFT), because two models may be practically indistinguishable from each other at low energies (low resolution) even if they differ significantly from each other at higher energies (finer resolution). This mathematical phenomenon is called **universality**, the subject of this article.

Universality can be a nuisance, because it restricts how much we can learn about nature from experiments that have limited energy. Universality is also beneficial, though, because it means that we can often construct relatively simple but accurate models of low-energy phenomena without including all of the details of higher-energy phenomena.

This article emphasizes another benefit of universality. In QFT, spacetime is often treated as a lattice for the purpose of constructing models nonperturbatively. In that case, we only care about the model's predictions at resolutions much coarser than the lattice scale, because the lattice is artificial. Thanks to universality, those low-energy predictions may be practically unaffected by lattice artifacts.[2]

Much of what we currently know about quantum field theory in d-dimensional spacetime with $d \geq 3$ comes from analyzing models that are close to trivial fixed points,[3] because that's where most of our mathematical tools work best.[4] This article emphasizes concepts that also apply in the vicinity of *nontrivial fixed points*,[3,5] even though the examples given are all in the vicinity of a trivial fixed point.

[1] Heuristically, this is because finer resolution means discerning the effects of smaller translations in space, which requires using larger momenta (because the momentum observables are the generators of translations in space), which implies larger energy (because in a Lorentz-symmetric model, the energy of an isolated probe cannot be less than the magnitude of its momentum).

[2] Lüscher (1987), section 2

[3] This will be defined in section 4.

[4] Article 22212 introduces one of these tools.

[5] Section 1 in Giuliani *et al* (2021) briefly reviews why physicists are relatively confident that the generalized concepts are valid even though many of the calculations we wish we could do are still beyond the reach of known mathematical methods.

2 Conceptual challenges

To study universality, we need to be clear about what we mean when we say that two models are *practically indistinguishable* from each other at low energies. This presents two basic conceptual challenges:

- Given a correspondence between the two models' observables, how should we quantify whether differences in their predictions are *large* or *small*? Large or small compared to what?

- Even more abstractly, what if a correspondence between the two models' predictions is not specified? Each model makes a set of many predictions, nominally infinitely many. How should we compare two unordered sets of predictions?

These challenges can be addressed using the *renormalization group* concept that will be introduced in sections 3-7.

3 The reversible renormalization group

In quantum field theory, observables are associated with regions of spacetime.[6] Let $\Omega(R)$ denote the set of observables associated with the region R, and let M denote the model defined by this association. For now, suppose that spacetime is being treated as a smooth manifold. For any **scale factor** $\lambda > 0$, define λR to be the region consisting of the points with coordinates $(\lambda x_1,\, \lambda x_2, \ldots \lambda x_d)$ for all $(x_1,\, x_2, \ldots x_d) \in R$, where d is the number of dimensions of spacetime. We can define a new model M' whose sets $\Omega'(R)$ of local observables are defined by[7]

$$\Omega'(R) = \Omega(\lambda R). \tag{1}$$

These transformations constitute one version of the **renormalization group (RG)**. Intuitively, if the transformation (1) could be implemented in the real world, it could be used as a lens to "zoom out" ($\lambda > 1$) or "zoom in" ($\lambda < 1$) on the original model. Like a real lens, this would be useful for matching the features of interest to the resolution of our measuring devices (analogous to the size of the pixels in a digital camera). Zooming out ($\lambda > 1$) is often called flowing **toward the infrared (IR)**, and zooming in ($\lambda < 1$) is often called flowing **toward the ultraviolet (UV)**. This article considers only flows toward the infrared ($\lambda > 1$).

The RG defines a flow in the space of possible models: if we start with one model and gradually slide the factor λ away from 1, then the corresponding sequence of transformations (1) traces out a specific path in the space of possible models. These transformations are *reversible* in the sense that the transformation with scale factor λ can be un-done by applying the transformation with scale factor $1/\lambda$. I'll call (1) the **reversible** renormalization group to distinguish it from another version of the RG that will be defined in section 7.[8]

[6] Article 21916

[7] A scale transformation (1) is often called a **dilation** or **dilatation**. The version with the extra syllable allegedly predates the shortened version (`https://english.stackexchange.com/questions/160496`), but both are commonly used in physics today with no difference in meaning.

[8] *Group* normally implies reversibility, but the version defined in section 7 is not reversible. *Renormalization semigroup* would be a better name for that version, but the evolution of language is not reversible.

4 Scale symmetry and fixed points

A model whose predictions look the same at all scales is called **scale invariant**, and this property of the model is called **scale invariance** or **scale symmetry**.[9,10]

More precisely, scale symmetry is a statement about the relationship between observables and regions of spacetime. Let $\Omega(R)$ be the set of observables localized in a region R of spacetime. In a model with scale symmetry, if all distances and time intervals are multiplied by $\lambda > 0$, then there is a unitary transformation $U(\lambda)$ such that

$$\Omega(\lambda R) = U^{-1}(\lambda)\Omega(R)U(\lambda) \tag{2}$$

for all R. Models with this symmetry are exceptional: most models don't have this symmetry. The massless free scalar model is one that does.

Equations (1) and (2) imply that if a reversible RG transformation is applied to a scale-invariant model, then the resulting model is unitarily equivalent to the original one. In this case, the "path" traced by (1) doesn't go anywhere – it stays where it started. For this reason, scale-invariant models are called **fixed points** of the renormalization group.[11] The simplest examples of scale-invariant models are massless models with no interactions. They're called **trivial** fixed points[12] because they lack interactions. Other scale-invariant models are called **nontrivial** fixed points. Our knowledge of of nontrivial scale-invariant models in dimensions $d \geq 3$ is still relatively immature,[13,14] so this article focuses mostly on the vicinity of trivial fixed points – while still trying to describe many of the general concepts in ways that are expected to apply near arbitrary fixed points.

[9] Sometimes the word *scale* is replaced by *dilation* or *dilatation* (footnote 7).

[10] In QFT, models with scale symmetry tend to have an even larger symmetry called *conformal symmetry* (Nakayama (2015)), which includes the scale symmetry group as a subgroup (article 38111). Examples of exceptions are mentioned in Riva and Cardy (2005) and in Komargodski (2012).

[11] When the renormalization group is cast as a system of ordinary first-order differential equations for the parameters in the action (article 22212), fixed points are sometimes also called **critical points** (Sfondrini (2013)), but physicists often use that name for something related-but-different (section 23).

[12] Trivial fixed points are also called *gaussian* fixed points, a name that describes the form of the path integral.

[13] Cappelli *et al* (2017)

[14] In contrast, our knowledge of conformally-invariant models in $d = 2$ is voluminous (Di Francesco *et al* (1997)).

5 Operators with scaling dimensions

By definition, a scale-invariant model admits a group of unitary operators $U(\lambda)$, one for each $\lambda > 0$, satisfying the condition (2). The effect of such a scale transformation on some observables – or, more generally, some operators – is especially simple. Let $\mathcal{O}(x)$ be a collection of operators,[15] one for each point x in spacetime, that are related to each other by translations in spacetime.[16] In a scale-invariant model, some of these collections satisfy

$$U^{-1}(\lambda)\mathcal{O}(x)U(\lambda) = \lambda^{\Delta}\mathcal{O}(\lambda x) \tag{3}$$

for some real number Δ. Such an operator $\mathcal{O}(x)$ is said to have **scaling dimension** Δ. In the massless free scalar model in d-dimensional spacetime, the operator $\partial\phi$ (the derivative of the field)[17] has scaling dimension $d/2$.

This definition only makes sense at a fixed point (a scale-invariant model), because only then do unitary operators $U(\lambda)$ satisfying the condition (2) exist.[18]

[15] Here, I'm using the word *operators* in a broad sense. They may not actually be defined as ordinary operators on the Hilbert space in continuous spacetime, not even when smeared over a neighborhood of the point x, but they are objects whose n-point correlations functions are well-defined as long as the points are all distinct (article 23277). They are ordinary operators when spacetime is treated as a lattice, but equation (3) treats spacetime as a continuum. Section 20 will distill a useful insight from this apparent tension.

[16] This condition assumes that the model has translation symmetry: for any translation $x \to x+\delta x$, the observable-sets $\Omega(R)$ and $\Omega(R+\delta x)$ are related to each other by a unitary transformation: $\Omega(R+\delta x) = U^{-1}(\delta x)\Omega(R)U(\delta x)$. The required property of the collection of operators $\mathcal{O}(x)$ is then expressed as $\mathcal{O}(x+\delta x) = U^{-1}(\delta x)\mathcal{O}(x)U(\delta x)$.

[17] In the massless free scalar model, the field $\phi(x)$ itself is not a local observable, because the zero-momentum part must be excluded (article 37301).

[18] Footnote 21 in section 6 will highlight another way to define scaling dimensions.

6 Scaling dimensions and the cluster property

Consider a scale-invariant model, and let $\mathcal{O}_1(x)$ and $\mathcal{O}_2(x)$ be local operators with scaling dimensions Δ_1 and Δ_2, respectively. The unitary operators $U(\lambda)$ leave the vacuum state $|0\rangle$ invariant, as long as the scale symmetry is not spontaneously broken.[19] This invariance combined with equation (3) implies

$$\langle 0|\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|0\rangle = \lambda^{\Delta_1+\Delta_2}\langle 0|\mathcal{O}_1(\lambda x_1)\mathcal{O}_2(\lambda x_2)|0\rangle.$$

Translation symmetry and Lorentz symmetry imply that the left-hand side depends on x_1 and x_2 only through the combination $|x_1 - x_2|$, which is translation- and Lorentz-invariant.[20] Combine this with the preceding equation to infer that the two-point function has the form[21]

$$\langle 0|\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|0\rangle \propto \frac{1}{|x_1 - x_2|^{\Delta_1+\Delta_2}}. \tag{4}$$

According to the **cluster property**,[22] the two-point function must decrease with increasing distance between the two points, at least asymptotically. This implies that all scaling dimensions must be positive ($\Delta > 0$).[23]

If a model is not scale invariant, then its correlation functions may involve one or more fixed length scales. Example: for large $r \equiv |x_1 - x_2|$, a typical asymptotic form for a two-point function is $\sim r^{-\alpha}e^{-r/\xi}$ for some exponent $\alpha > 0$ and some **correlation length** $\xi > 0$. In a non-empty[24] scale-invariant model, the correlation length is infinite.

[19] Crewther (2020) reviews some studies of spontaneous breaking of scale symmetry. Cresswell-Hogg and Litim (2022) includes a brief annotated list of references. Coradeschi *et al* (2013) reviews part of the concept.

[20] $|x| \equiv \sqrt{\sum_{a,b}\eta_{ab}x^a x^b}$, where η_{ab} are the components of the Minkowski metric. After Wick rotation to euclidean spacetime (section 9), this becomes $|x| \equiv \sqrt{x \cdot x}$ where $x \cdot x$ is the usual dot product.

[21] When the $\mathcal{O}(x)$ are not really operators (footnote 15), we can use (4) to define scaling dimensions.

[22] Article 21916

[23] This is mentioned in the text above equation (21.51) in Fradkin (2021) (and in the online version Fradkin (2022b)). In models that have conformal symmetry, the result $\Delta > 0$ can also be derived from that larger symmetry combined with the positivity of the Hilbert-space inner product (Simmons-Duffin (2016), section 7.3).

[24] *Empty* will be defined in section 10.

7 The irreversible renormalization "group"

If we don't care about high-resolution observables, then we might as well exclude them from the model. This is roughly the same[25] as imposing a **high-energy cutoff**, also called an **ultraviolet (UV) cutoff**. When we treat spacetime as a lattice, we're automatically imposing a UV cutoff $\Lambda_0 \sim 1/\epsilon$, where ϵ is the distance between neighboring lattice sites.[26,27] Making Λ_0 larger corresponds to making the lattice finer. This section defines a modified RG transformation that preserves the magnitude of the UV cutoff.

Let $\Omega(R, \Lambda_0)$ denote the set of observables associated with a region R of spacetime in a model M with a UV cutoff at the scale Λ_0. For any $\lambda > 1$, we can define a new model M' using this two-step process:[28]

1. First, discard all observables involving energies $> \Lambda_0/\lambda$, which effectively reduces the value of the cutoff from Λ_0 to Λ_0/λ.

2. Then, apply equation (1) to move the cutoff back to its original value Λ_0.

The first step is defined only for $\lambda > 1$. The second step is reversible, but the first step is not. I'll call this set of transformations the **irreversible renormalization group**.[29]

[25]These two concepts are not quite the same, because when spacetime is treated as a lattice, observables may be defined at individual points even though the lattice provides a high-energy cutoff. Footnote 1 in section 1 ignored this possibility because it's artificial.

[26]This article uses the euclidean version of the path integral (section 9), which replaces Lorentz symmetry in d-dimensional spacetime with ordinary rotation symmetry in d-dimensional euclidean space. In this context, distance and duration are synonymous, and so are energy and momentum.

[27]If the number of lattice sites is finite, then we're also imposing an **infrared (IR) cutoff**, which makes the spectrum of momenta (and energy) discrete instead of continuous.

[28]If an IR cutoff is also present (footnote 27), then a third step should be added to preserve the value of that cutoff, too, because otherwise the whole finite-volume spacetime would shrink toward zero size when flowing toward the IR. In this article, that third step is left implicit.

[29]This is usually just called the *renormalization group*, without the adjective *irreversible*, and it's usually described in terms of coefficients in the lagrangian instead of observables (example: Zomolodchikov (1986)).

8 Another type of irreversibility

The words *reversible* and *irreversible* were used in sections 3 and 7 to indicate whether an individual finite RG transformation can be un-done. The word *irreversible* is also used for a different concept, one that applies to the pattern of RG flows that go from one fixed point to another (asymptotically).[30]

Suppose that a flow toward the IR exists that goes from an arbitrarily small neighborhood of one fixed point M_1 to an arbitrarily small neighborhood of another fixed point M_2. We might wonder whether another flow toward the IR could also exist, one that goes from M_2 back to M_1. If that's not possible, then the pattern of flows toward the IR is called **irreversible**.[31]

Among models satisfying a few standard conditions, this type of irreversibility has been proved for fixed points with conformal symmetry,[32] but only for special values of d, the number of spacetime dimensions. The result is called the **c-theorem** for $d = 2$[33] and the **a-theorem** for $d = 4$.[34] A similar result called the **F-theorem** is conjectured for $d = 3$.[35,36] The introduction in Jensen and O'Bannon (2016) reviews the idea, goals, and status of theorems like these.

The rest of this article uses the word *irreversible* as it was used in section 7, not the way it was used in this section.

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[30]This article doesn't consider flows with limit cycles (Glazek and Wilson (2002)) or chaotic flows (Damgaard and Thorleifsson (1991)). Curtright *et al* (2012) comments on the (in)significance of the *irreversibility* described in this section for those types of flows.

[31]This should not be confused with the different type of (ir)reversibility that was described in sections 3 and 7.

[32]In quantum field theory, models with conformal symmetry are called *conformal field theories (CFTs).*

[33]Zomolodchikov (1986)

[34]Komargodski and Schwimmer (2011), Komargodski (2011)

[35]The status of this conjecture is reviewed in Nakayama (2015), section 9.2.1.

[36]Section 1 in Pufu (2016) says, "the RG flow is not reversible: if there exists a relativistic RG flow between CFT1 in the UV and CFT2 in the IR, a relativistic RG flow between CFT2 in the UV and CFT1 in the IR is ruled out [in $d = 3$ by the F-theorem]."

9 The models used in this article

Consider a model involving only a single scalar field[37] in d-dimensional spacetime, which will be treated as a lattice, and let I be some time-ordered product of field operators and their expectation values.[38] This article focuses on models in which the vacuum expectation value of the operator I can be reconstructed from the **euclidean path integral**[39]

$$\langle I\rangle \equiv \frac{\int [d\phi]\ e^{-S[\phi]} I[\phi]}{\int [d\phi]\ e^{-S[\phi]}} \tag{5}$$

using Wick rotation.[40] This includes models that are effectively Lorentz symmetric at resolutions much coarser than the lattice scale. The insertion $I[\phi]$ (which represents the operator I) and the euclidean action $S[\phi]$ are both expressed in terms of the scalar field variables $\phi(x)$ and their discretized derivatives with respect to x.

The path integral can also be expressed in terms of the lorentzian action $S_L[\phi]$, using factors of $e^{iS_L[\phi]}$ instead of $e^{-S[\phi]}$, but only if the initial and final states are also included as explicit factors in the integrand. One advantage of using the euclidean version (5) is that the initial and final states don't need to be included explicitly: the path integral (5) automatically selects the vacuum expectation value. For that reason, this article uses the euclidean version (5), and the euclidean action $S[\phi]$ will just be called the **action**.

For the examples in this article, the action is proportional to $\sum_x L(x)$, where the sum is over all points in spacetime, and the lagrangian $L(x)$ is proportional to $(\partial\phi(x))^2 + V(\phi(x))$ with $V(-\phi) = V(\phi)$.

[37] The generalization to multiple scalar fields is straightforward.

[38] Mnemonic: I stands for *insertion* or *integrand*, because of the way this operator is represented in the path integral (5).

[39] Article 63548

[40] If the result of evaluating the path integral is expressed as a function of the time-step dt, then the vacuum expectation value of I is recovered by replacing every occurrence of dt with $i\,dt$ (if the fields are all scalar fields). The name **Wick rotation** refers to writing this as $e^{i\theta}dt$ and then "rotating" θ continuously from 0 to $\pi/2$.

10 Continuum limits

Consider a model of the type described in section 9, treating spacetime as a lattice so that the path integral is well-defined. Adjustable parameters in the action are called **bare parameters**. In a model with realistic applications, we would tune the values of the bare parameters to make the model's predictions line up with the available measured data. We are normally only interested in the model's predictions at resolutions much coarser than the lattice scale, because the lattice is artificial. This condition can be expressed as $\Lambda \ll \Lambda_0$, where Λ_0 is the UV cutoff and Λ is an energy scale that characterizes the finest resolution of the measurements of interest.

Taking a **continuum limit** (a limit in which spacetime becomes continuous) requires pushing the ratio Λ/Λ_0 all the way to zero. We can think of this either as taking Λ to zero in units of Λ_0 or as taking Λ_0 to infinity in units of Λ. The simplest type of continuum limit keeps the values of the bare parameters fixed while taking Λ to zero in units of Λ_0. This is equivalent to flowing toward the IR under the renormalization group, all the way to the flow's asymptotic end. The result depends on the values of the bare parameters:

- The model may become empty, in the sense that the only state(s) whose energy remains finite in physically meaningful units are the vacuum state(s) (section 11). Such a model has no useful content, not even as a toy model.[41]

- The flow may asymptotically approach a scale-invariant model with nonzero correlations (section 12). The set of models that asymptotically approach a given fixed point is called the **basin of attraction** for that fixed point.[42]

To approach a model with nonzero correlations but without scale symmetry, we need to vary the bare parameters while taking the limit (section 13). Such a limit cuts across RG flows instead of following a single RG flow.

[41] Models with only vacuum state(s) can be instructive toy models when spacetime has a nontrivial topology. That subject is called **topological quantum field theory (TQFT)**.

[42] Banks (2008), section 9.4 , pages 152-153

11 An empty continuum limit

This section describes an example of the first possibility that was listed in section 10. Consider the action

$$S[\phi] = \frac{1}{2}\epsilon^d \sum_x \Big(\big(\partial\phi(x)\big)^2 + m^2\phi^2(x)\Big) + \text{constant} \tag{6}$$

with $m \geq 0$ and a constant term that may depend on the lattice spacing ϵ but not on ϕ. Both terms are positive because this is the *euclidean* action (section 9). Using this action in the path integral (5) gives the free scalar model.[43] The parameter m turns out to be the mass of a single particle,[44] and the correlation length is proportional to $1/m$.

The mass is finite in units of the UV cutoff $\Lambda_0 \sim 1/\epsilon$, but in a continuum limit, the lattice scale Λ_0 becomes infinite in units of any physically meaningful scale Λ. If we take the limit by following the flow of the renormalization group (section (7)), then m/Λ_0 remains fixed. That implies $m/\Lambda \to \infty$, so the correlation length goes to zero in physically meaningful units, and the energies of all non-vacuum states become infinite in physically meaningful units,[45] so the model becomes empty (as defined in section 10). That's not what we want.

[43] A model is called *free* if its equations of motion are linear in the fields, which implies that its particles don't interact with each other (article 30983).

[44] Article 30983

[45] Article 00980 shows that the lowest non-vacuum energy is $\sim m$.

12 A continuum limit with scale symmetry

To prevent the model from becoming empty in the limit defined by the RG flow, we can set $m = 0$. If we didn't make any other changes, then the model would be undefined, because the path integral includes an integral over every field variable $\phi(x)$, but the action (6) with $m = 0$ is independent of the combination $\sum_x \phi(x)$. To fix this problem, we can use the action

$$S[\phi] = \epsilon^d \sum_x \frac{1}{2}\big(\partial\phi(x)\big)^2 + c\left(\epsilon^d \sum_x \phi(x)\right)^2 + \text{constant} \tag{7}$$

with $c > 0$. With this action, if the RG flow defined in section 7 is followed to its asymptotic end, the result is a model with nonzero correlations over arbitrarily large distances in physically meaningful units. The model is scale invariant except for the UV cutoff (and the IR cutoff), which can subsequently be removed. This is the massless free scalar model.[46]

The $(\sum_x \phi(x))^2$ term might seem awkward because it's not local: it involves products $\phi(x)\phi(y)$ with arbitrarily large $|x-y|$. When $d \geq 4$, a less obvious option is available:[47] we can use a local action of the form

$$S[\phi] = \epsilon^d \sum_x \left(\frac{1}{2}\big(\partial\phi(x)\big)^2 + c_2\phi^2(x) + c_4\phi^4(x)\right) + \text{constant} \tag{8}$$

with $c_4 > 0$ and with c_2 tuned to a special negative value (called the **critical point**)[48] that makes the correlation length infinite in units of ϵ. The model with this specially-tuned action flows to the same trivial fixed point in the IR as the model with action (7), but only if $d \geq 4$. If $d \leq 3$, then c_2 and c_4 can be chosen so that the model flows to a different fixed point (article 79649).

[46]More precisely, this is a version of the massless free scalar model in which the zero-momentum part of $\phi(x)$ is excluded and the zero-momentum part of $\dot{\phi}(x)$ is arbitrary (article 37301, and section 6.3.3 in Di Francesco *et al* (1997)).

[47]Section 15

[48]Section 23, foreshadowed in footnote 11 in section 4

13 A continuum limit without scale symmetry

To get a model that has correlations over nonzero distances but doesn't have scale symmetry, we need to take a different kind of continuum limit. Instead of following an RG flow, we need to cut across RG flows – we need to make the bare parameters functions of Λ/Λ_0 so that they vary while taking $\Lambda/\Lambda_0 \to 0$. This is called **renormalization**.

For a simple example,[49] consider the action (6) again, but now keep the value of m/Λ fixed while taking the limit $\Lambda/\Lambda_0 \to 0$. The result is the free scalar model with a single-particle mass m that is nonzero but finite in units of the physically meaningful scale Λ. In units of the lattice scale Λ_0, the correlation length becomes infinite (m/Λ_0 goes to zero), but it remains finite in units of Λ.

[49] The name *renormalization* is used mainly when the action involves higher-than-quadratic terms, and usually not for quadratic cases like this one, even though the basic idea is the same.

14 Universality

Let $S_*[\phi]$ be any action in the basin of attraction of a given fixed point, and let

$$S[\phi] = S_*[\phi] + \delta S[\phi] \tag{9}$$

be another action that differs from $S_*[\phi]$ by a small perturbation $\delta S[\phi]$. If the new action $S[\phi]$ still belongs to the same fixed point's basin of attraction, then the perturbation $\delta S[\phi]$ is called **irrelevant**. An irrelevant perturbation doesn't change the ultimate destination of the RG flow.

In contrast, a **relevant** perturbation pushes the model out of that fixed point's basin of attraction.[50] In this case, the RG flow eventually carries the model (9) away from the original fixed point. It might start to approach the original fixed point initially, but eventually it turns away.[51,52] **Universality** refers to the fact that for a typical fixed point, the number of linearly independent relevant perturbations (modulo the irrelevant ones) is relatively small, at any given point on the fixed point's basin of attraction.[53] This refines the rough definition of *universality* that was given in section 1.

Relevant perturbations are the ones whose coefficients need to be tuned as functions of Λ/Λ_0 in order to get a continuum limit without scale symmetry,[54] at least if we want the resulting model to be "close to" the original fixed point. If we apply a relevant perturbation and then follow the RG flow to its end without that kind of tuning, then the model might become empty (as in the example in section 11) or it might approach a different fixed point without becoming empty.[55]

[50] This assumes that the perturbation is small. Large perturbations may interact with each other in ways that change the effects that they would have had individually.

[51] Banks (2008), page 153, text following figure 9.3

[52] It may ultimately approach a different fixed point instead, so a perturbation that is relevant with respect to one fixed point may be irrelevant with respect to a different fixed point.

[53] This last qualification is needed because the set of (ir)relevant perturbations varies throughout the basin of attraction. The precise correspondence between an operator's (ir)relevance and its scaling dimension (section 18) holds only very close to the fixed point itself. Elsewhere in the basin of attraction, that correspondence is modified (unless the definition of *scaling dimension* is correspondingly generalized, as some authors do).

[54] Lüscher and Weisz (1987), below equation (9.1)

[55] Examples of this are harder to find – not because they don't exist, but because the math is more difficult.

15 Examples

The action (7) is in the basin of attraction of a trivial fixed point. We can use that action as an example of the S_* in equation (9). In that case, the perturbation

$$\delta S[\phi] \propto \sum_x \phi^2(x) \tag{10}$$

is *relevant*.[56] This was explained in sections 11-13.

A more interesting example is a perturbation of the form

$$\delta S[\phi] \propto \sum_x \big(\phi^4(x) + \gamma\phi^2(x)\big) \tag{11}$$

with constant γ.[57] By tuning the parameter γ to a special negative value, we can make the correlation length infinite, and then this perturbation is *irrelevant* when $d \geq 4$. The required value of γ depends both on d and on the perturbation's overall coefficient.[58] Section 16 will describe this in more detail for $d = 4$.

A perturbation of the same form (11) is *relevant* when $d = 2$ or $d = 3$, even if the value of γ is tuned to make the correlation length infinite. For that special value of γ, the perturbed action is in the basin of attraction of a different fixed point – not the one that was described in section 12. Sections 17 will describe this in more detail for $d = 2$, and article 79649 highlights the case $d = 3$.

Section 19 will relate these examples to scaling dimensions.[59]

[56]For the path integral (9) to be well-defined, the coefficient of the perturbation (10) must be positive, and then the path integral (5) remains well-defined even if the second term is (7) is omitted.

[57]Here, *constant* means independent of the spacetime point x and also independent of the field variables $\phi(x)$.

[58]The overall coefficient of the perturbation (11) must be positive, and then the path integral (5) remains well-defined even if the second term is (7) is omitted.

[59]Scaling dimensions were defined in section 5.

16 The irrelevance of (11) when $d = 4$

Lüscher and Weisz (1987) studied the action[60]

$$S = \epsilon^d \sum_x \left(\frac{1}{2}\big(\partial\phi(x)\big)^2 + \frac{g_2}{2\epsilon^2}\phi^2(x) + \frac{g_4}{4!\epsilon^{4-d}}\phi^4(x) \right) \tag{12}$$

for $d = 4$. According to their Table 1, for a given value of g_4, making the correlation length infinite requires tuning g_2 to the value shown here:[61]

g_4	g_2	$-g_2/g_4$
0.000	0.000	
1.313	−0.100	0.076
2.693	−0.202	0.075
4.136	−0.314	0.076
5.654	−0.422	0.075
7.247	−0.535	0.074
8.901	−0.656	0.074
10.646	−0.775	0.073
12.472	−0.899	0.072
14.380	−1.026	0.071

g_4	g_2	$-g_2/g_4$
16.376	−1.157	0.071
41.970	−2.717	0.065
81.827	−4.839	0.059
144.867	−7.795	0.054
246.674	−12.047	0.049
418.478	−18.422	0.044
728.938	−28.637	0.039
1361.594	−47.055	0.035
3056.380	−91.087	0.030
$\to \infty$	$\to -\infty$	$\to 0.025$

When the correlation length is infinite, taking the straightforward continuum limit (with no additional parameter-tuning along the way) gives a scale-invariant model, and in this case that fixed point must be trivial because nontrivial fixed points don't exist for this class of models in $d = 4$.[62]

Contrary to what many texts might seem to say, the ϕ^4 term by itself is *not* an

[60]This is equation (2.2) in Lüscher and Weisz (1987). Their ϕ_0 is my $\phi \times \epsilon$, their g is my g_4, and their m_0^2 is my g_2. The traditional notation m_0^2 can be misleading, because if $g_4 > 0$, then m_0^2 must be *negative* to make the correlation length much larger than the lattice spacing ϵ, as shown here in the table.

[61]This table was derived from their Table 1 by using their equations (2.1)-(2.4).

[62]Section 22

irrelevant perturbation.[63] The irrelevant perturbation is

$$\phi^4 + \gamma\phi^2 + \text{constant} \tag{13}$$

for a particular negative value of γ. The results tabulated above show that the required value of γ depends on the value of g_4, apparently[64] approaching

$$\gamma \approx -0.076 \times \frac{4!}{2\epsilon^2} \tag{14}$$

as $g_4 \to 0$.

Analogous results for $d = 2$ are shown in Schaich and Loinaz (2009), but in that case the corresponding perturbation (13) is relevant with respect to the fixed point $g_4 = g_2 = 0$, so the straightforward continuum limit leads to a different fixed point. This will be described in section 17.

[63] When people say that a ϕ^4 term is irrelevant in $d \geq 4$, they're really referring to what is sometimes called the "renormalized ϕ^4" term (example: Serone (2018), section 5.9, text between equations (5.9.1)-(5.9.2)), as described more explicitly in articles 22212 and 23277. The main message in this section is that the "renormalized ϕ^4" term has the form shown in equation (13), with a negative value of γ.

[64] The value of γ in the limit $g_4 \to 0$ was not given explicitly in Lüscher and Weisz (1987), as far as I noticed. The estimate (14) is based on a simplistic linear extrapolation of the tabulated results for $g_4 \lesssim 5$, motivated by the fact that the ratio g_2/g_4 is nearly constant for those entries in the table.

17 The relevance of (11) when $d = 2$

In $d = 2$, the ϕ^4 model is in the same universality class as the Ising model,[65] which means that after their parameters are tuned to make the correlation length infinite, both models flow toward the same scale-invariant model in the IR. This doesn't contradict the fact that (11) is a relevant perturbation. The perturbation is relevant with respect to one scale-invariant model (the one with action (7)), but flowing toward the IR leads to a different scale-invariant model M. The model M is nontrivial.[66] If it were trivial, then its four-point function could be reduced to a sum of products of two-point functions,[67] but the four-point function for the basic field $\sigma(x)$ in the model M is[68]

$$\big\langle\sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\big\rangle^2 \propto f(1,2\,|\,3,4) + f(1,3\,|\,2,4) + f(1,4\,|\,2,3)$$

with

$$f(1,2\,|\,3,4) \equiv \sqrt{\frac{|x_1-x_2|\,|x_3-x_4|}{|x_1-x_3|\,|x_1-x_4|\,|x_2-x_3|\,|x_2-x_4|}}.$$

The model M is "equivalent" to a free Majorana fermion field in a computationally useful sense,[69] but the "equivalence" is nonlocal,[70] so this does not contradict the assertion that M is nontrivial.[71]

[65]Section 1 in Serone *et al* (2018), and and section 1 in Serone *et al* (2019)

[66]Section 6.3 in Aizenman (2021) calls it *non-gaussian.*

[67]Article 22212

[68]This this is the case $n = 2$ of equation (2.23a) in Di Francesco *et al* (1987), or equation (12.63) in Di Francesco *et al* (1997). Here, each x_k is a point in 2-dimensional euclidean spacetime: $x_k = ((x_k)_1,\, (x_k)_2)$.

[69]The relationship of M to the free Majorana fermion field is reviewed in Molignini (2013), especially equations (3.12), (3.14), (3.28), (3.31). It is also reviewed more briefly in in sections 21.6.2 and 21.6.3 of Fradkin (2021) (and the online version Fradkin (2022b)). The reslationship is explored further in Ardonne and Sierra (2010).

[70]Di Francesco *et al* (1987)

[71]In QFT, a model is defined by the relationship between observables and regions of spacetime. The "equivalence" between M and the free Majorana fermion field does not respect that relationship: it is a more relaxed kind of equivalence.

18 Universality and scaling dimensions

This section presents a general intuitive argument that leads to a simple correspondence between the (ir)relevance of a perturbation and its scaling dimension.[72] This correspondence works well near the fixed point that is used to define the scaling dimensions, except in cases where the correspondence says that the perturbation is *marginal* – the threshold between relevant and irrelevant. In such cases, the perturbation usually turns out to be either slightly relevant or slightly irrelevant, and determining which way the balance tips requires a more detailed analysis.[73] Section 20 will explain why the intuition described here is not perfect, which helps explain why the resulting correspondence doesn't resolve nearly-marginal cases.

Let M_* be a scale-invariant model (also called a *fixed point*, as in section 4), let $S_*[\phi]$ be an action in the basin of attraction for M_*, let $\mathcal{O}(x)$ be an operator that has scaling dimension Δ at that fixed point (section 5), and consider the modified action

$$S[\phi] = S_*[\phi] + \epsilon^d \sum_x c\,\mathcal{O}(x), \tag{15}$$

with coefficient c. Let $I(x_1, ..., x_k) \equiv \mathcal{O}_1(x_1)\cdots\mathcal{O}_k(x_k)$ be a product of operators that satisfy (3) in the model with action S_*. In the model with action (15), expectation values may be written[74,75]

$$\big\langle I(x_1, ..., x_k)\big\rangle = \left\langle I(x_1, ..., x_k)\exp\left(-\int d^dx\;c\,\mathcal{O}(x)\right)\right\rangle_* \tag{16}$$

where $\langle\cdots\rangle_*$ is the expectation value in the model with action S_*. Equation (16) is the starting point for a computational method called **conformal perturbation**

[72] The concept of *scaling dimension* was introduced in section 5.

[73] Article 22212

[74] Section 9

[75] The unbounded integral $\int d^dx\;c\,\mathcal{O}(x)$ can be handled by temporarily replacing the constant c with a function $c(x)$ of compact support and then taking the limit $c(x)\to$ constant after evaluating the expectation values.

theory.[76]

If we could treat $\mathcal{O}(x)$ as if it were an isolated insertion and could treat space-time as an infinite continuum, then the right-hand would be invariant under (3), so

$$\big\langle I(x_1,...,x_k)\big\rangle \propto \left\langle I(\lambda x_1,...,\lambda x_k)\exp\left(-\int d^dx\; c\,\lambda^{\Delta}\mathcal{O}(\lambda x)\right)\right\rangle_* . \tag{17}$$

By changing the integration variable, this may also be written

$$\big\langle I(x_1,...,x_k)\big\rangle \propto \left\langle I(\lambda x_1,...,\lambda x_k)\exp\left(-\int d^dx\; c\,\lambda^{\Delta-d}\mathcal{O}(x)\right)\right\rangle_* . \tag{18}$$

This shows that in the model with action (15), if we could treat $\mathcal{O}(x)$ as if it were isolated, then a scale transformation $x \to \lambda x$ combined with the replacement $c \to c\,\lambda^{\Delta-d}$ would leave the correlation function invariant up to an overall λ-dependent factor, and then the effect of a renormalization group transformation (1) would be equivalent to replacing $c \to c\,\lambda^{d-\Delta}$. Flowing toward the IR corresponds to $\lambda > 1$, so we have deduced this correspondence:[77]

> If the magnitude of c is small enough, then the last term in (15) is a relevant perturbation if $\Delta < d$ and is irrelevant if $\Delta > d$. If $\Delta = d$, then a more detailed analysis is needed, because the argument outlined above is not perfect (section 20).

Scale-invariant models tend to have only a small number of linearly independent operators with $\Delta < d$.[78] The correspondence highlighted above then says that the number of independent relevant perturbations is also small. That's universality.

[76] Example: Sen and Tachikawa (2017) uses conformal perturbation theory to study whether perturbations by marginal operators preserve conformal invariance when $d \geq 3$.

[77] Section 20 will mention why the condition *small enough* is needed.

[78] The text below equation (135) in Simmons-Duffin (2016) relates this to the assumption that the operator $e^{-\beta D}$ is trace-class, where D is the dilation operator. Page 9 in Hollowood (2009) says "CFTs only have a finite (and usually small) number of relevant couplings" but doesn't prove this statement or qualify its scope.

19 Examples

To illustrate the correspondence highlighted in section 18, here are the examples from section 15 again,[79] this time expressed in terms of operators[80] that satisfy the condition (4):

- For an appropriate function r of the lattice spacing,[81] the operator

$$\mathcal{O}(x) \equiv \phi^2(x) - r$$

  has scaling dimension $\Delta = d - 2$ after taking the continuum limit to the trivial fixed point. This is the *normal-ordered* version of $\phi^2(x)$.

- For that same function r, the operator

$$\mathcal{O}(x) \equiv \phi^4(x) - 6r\phi^2(x) + 3r^2 \tag{19}$$

  has scaling dimension $\Delta = 2(d - 2)$ after taking the continuum limit to the trivial fixed point. This is the *normal-ordered* version of $\phi^4(x)$.

[79] The constant terms included here were omitted in section 15.

[80] Recall footnote 15 in section 5: When spacetime is treated as a lattice, the examples described in this section are defined as operators on the Hilbert space, but that is no longer true in in the continuum limit, not even when they are smeared over a spacetime region of arbitrarily small nonzero size (article 23277). In the continuum limit, they become a more general type of object for which the vacuum expectation value of the product $\mathcal{O}(x_1)\mathcal{O}(x_2)\cdots\mathcal{O}(x_n)$ is well-defined as long as the points x_k are all distinct from each other. These objects still have scaling dimensions, but their scaling dimensions are defined by equation (4) instead of by equation (3).

[81] Article 23277

20 Why the intuition isn't perfect

Section 18 used a heuristic argument to deduce a correspondence between the (ir)relevance of a term in the action and the scaling dimension of the corresponding operator. This section explains why the argument is not perfect.

The argument in section 18 relies on equation (3). That equation assumes that if x is a point in spacetime, then λx is also a point in spacetime for any $\lambda > 0$. That requires spacetime to have infinite size, but then the integral $\int d^d x\ \mathcal{O}(x)$ is not defined. We can fix this by using $\int d^d x\ f(x)\mathcal{O}(x)$ instead, where $f(x)$ is nonzero only within some finite but arbitrarily large radius r. This was acknowledged in footnote 75 in section 18.

Equation (3) also assumes that spacetime is continuous. That's a problem, because then $\mathcal{O}(x)$ is not an ordinary operator on the Hilbert space: if $|0\rangle$ is the vacuum state, then the norm of $\mathcal{O}(x)|0\rangle$ is undefined.[82] For most of the $\mathcal{O}(x)$ that we might want to consider, smearing doesn't help: the norm of $\int d^d x\ f(x)\mathcal{O}(x)|0\rangle$ is typically still undefined.[83] To make sense out of the right-hand side of (16), some kind of UV cutoff – like treating spacetime as a lattice – is essential.[84]

We can still use an approximate version of equation (3). Without the last term in (15), the model would be scale invariant at sufficiently low resolution except for artifacts due to the finite value of ϵ and the finite overall size of the lattice. If those two scales are separated by a large enough factor, say $10^{10\,000\,000}$, then the deviation from exact scale invariance would be negligible over a large intermediate range of scales, which we can take to represent the resolution of our measurements. In that context, and using the abbreviation

$$\int d^d x\ \cdots \equiv \epsilon^d \sum_x \cdots\ ,$$

[82] Footnote 15 in section 5, and footnote 80 in section 19

[83] Article 23277

[84] Fradkin (2021) acknowledges this in the text below equation (15.118) (and in the online version Fradkin (2022)). His equation (15.113) corresponds to this article's equation (16), and his equation (15.114) corresponds to this article's equation (18), except that he only writes the partition function instead of a correlation function.

a group of unitary operators $U(\lambda)$ should exist for which

$$U^{-1}(\lambda)\left(\int d^dx\ f(x)\mathcal{O}(x)\right)U(\lambda) \approx \lambda^\Delta \int d^dx\ \lambda^{-d}f(x/\lambda)\mathcal{O}(x) \tag{20}$$

for any a smooth smearing function $f(x)$ that may have compact support but that is practically constant over distances comparable to ϵ. Equation (20) is an approximate version of (3) because if the right-hand side of (20) were defined in continuous spacetime, then the substitution $x = \lambda x'$ could be used to write it as

$$\lambda^\Delta \int d^dx'\ f(x')\mathcal{O}(\lambda x'). \tag{21}$$

Unlike (21), though, the right-hand side of (20) makes sense even when x is restricted to points of the lattice, which are the points where the operator $\mathcal{O}(x)$ is defined. The approximate equation (20) can be used in place of equation (3) to get an approximate version of (18).

The key message here is that argument presented in section 18 can be modified to account for the UV cutoff, but this can also modify its conclusion about the rate at which the effect of a given perturbation becomes more or less important as λ increases. In particular, a perturbation with a marginal scaling dimension $\Delta = d$ is not necessarily an exactly marginal perturbation. Section 15.6 in Fradkin (2021) (and the online version Fradkin (2022)) uses the operator product expansion[85] to explain how the UV cutoff gives rise to higher-order corrections that can make perturbations with $\Delta = d$ either slightly relevant or slightly irrelevant. Article 22212 uses a different approach to reach the same conclusion.

One of the key insights from those analyses is that after accounting for the UV cutoff, the derivation of the correspondence in section 18 implicitly relies on a small-coupling approximation. As a result, scaling dimensions don't perfectly predict which perturbations are to relevant/irrelevant, except close to the fixed point.

[85] Article 23277

21 Marginal and marginally (ir)relevant terms

Starting with the action S_* of a scale-invariant model, a perturbation $\mathcal{O}$ is called **(exactly) marginal** if taking the straightforward continuum limit of the model with action (15) would give a scale-invariant model that differs from the original one.[86,87] *Marginal* is the borderline case between *relevant* and *irrelevant*.[88]

As explained in section 20, the correspondence highlighted in section 18 is only valid to first order in an expansion in powers of the coefficient c, so a perturbation with $\Delta = d$ may turn out to be either irrelevant or relevant when analyzed more carefully. Article 22212 shows that when $d = 4$, the operator (19) is irrelevant even though its scaling dimension is $\Delta = d$. It is often called **marginally irrelevant**, because it is so close to the threshold that its (ir)relevance cannot be inferred from its scaling dimension alone. It is irrelevant, but just barely.

[86] Perturbations with scaling dimension $\Delta = d$ don't necessarily satisfy this condition. Many authors use the short name *marginal* for any perturbation with $\Delta = d$ and use the longer name *exactly marginal* for those that satisfy the stronger condition described here.

[87] A continuum of scale-invariant models related to each other by such perturbations is called a **conformal manifold** (Baggio *et al* (2018)).

[88] According to section 1.1 in Gerchkovitz *et al* (2016), "exactly marginal operators are common in supersymmetric theories in $2 \le d \le 4$" (d is the number of spacetime dimensions). For $d \ge 3$, supersymmetric models provide the only known examples of exactly marginal perturbations (Baggio *et al* (2018), first paragraph in section 1).

22 The importance of marginally irrelevant terms

Models with an action of the form (8) are called $\boldsymbol{\phi}^4$ **models**. Within this family of models, nontrivial strict continuum limits do exist if $d \leq 3$,[89] but not if $d \geq 4$,[90] at least not if the $\phi \to -\phi$ symmetry is not spontaneously broken.[91]

Marginally irrelevant perturbations are important because even though their effects would vanish if the RG flow were followed to its end (toward the IR), they might vanish only very slowly compared to the rate at which Λ/Λ_0 goes to zero. For that reason, marginally irrelevant perturbations can have significant effects at resolutions much coarser than the lattice scale even if a nontrivial strict continuum limit doesn't exist. Quantum electrodynamics (QED) is probably one of those models: it apparently does not have a nontrivial strict continuum limit, but a significant interaction is still present over a large range of energies (resolutions) for which the deviation from continuous spacetime is negligible. This range of energies is enough to cover all of the model's intended applications.[92]

[89]The case $d = 3$ is addressed in literature about the **Wilson-Fisher fixed point**, like section 5.3.4 in Skinner (2016) and the text below equation (15.75) in Fradkin (2021) (also the online version Fradkin (2022)). The case $d = 2$ is addressed in the vast literature about two-dimensional conformal field theory.

[90]For $d \geq 5$, this is proved in Aizenman (1981) and Aizenman (1982). Section 6.4 in Duminil-Copin (2022) reviews the idea of the proof. Aizenman (2021) reviews the idea in more detail. The proof is extended to $d = 4$ in Jora (2015) and Aizenman and Duminil-Copin (2019), and section 1 in Podolsky (2010) reviews some history.

[91]This condition is mentioned in the text below equation (13.3) in Aizenman (1982).

[92]According to section 2.2 in McGreevy (2019), "Landau and many other smart people gave up on QFT as a whole because of this silly fantasy about QED in an unphysical regime." Based on its context, I interpret this quote to mean that those people initially hoped that QED would have a nontrivial strict continuum limit and that they gave up on QFT when they realized that it doesn't, even though QED still has a perfectly good non-perturbative construction – treating spacetime as a lattice – with a large range of applications in which artifacts of the lattice are utterly negligible.

23 The critical manifold

If a fixed point is not the empty model, then its basin of attraction is also called its **critical manifold**.[93] The name *basin of attraction* alludes to the fact the RG flow carries these models to the given fixed point. The name *critical manifold* alludes to the fact that each of these models has infinite correlation length,[94] as it must in order for the RG flow to approach a non-empty model asymptotically.[95,96] This is consistent with the way the adjective *critical* is used in statistical physics: a point in parameter-space at which a correlation length diverges is called a **critical point**. A famous example is the phase diagram of a fluid like water or carbon dioxide, which has an isolated critical point in the pressure-temperature plane.[97] Right at the critical point, the fluid becomes semi-opaque, a phenomenon called **critical opalescence**.

Points on the critical manifold are often called **critical points**, including the fixed point itself (if it's not an empty model). A fixed point that admits more than one linearly independent relevant perturbation (modulo the irrelevant ones) is called a **multicritical** fixed point.[98,99] In other words, a multicritical fixed point is one for which more than one parameter must be tuned as a function of Λ/Λ_0 in order to approach the fixed point as $\Lambda/\Lambda_0 \to 0$.

[93] Banks (2008), section 9.4 , pages 152-153 (also Wu and Car (2019))

[94] Wilson and Kogut (1974), page 161, text above figure 12.1

[95] Here, I'm using the name *correlation length* to mean any length scale, other than the lattice spacing ϵ, that characterizes form of a two-point correlation function as a function of the distance between the two points, as the distance goes to infinity. Section 6 described one example.

[96] I'm only considering continuum limits for which the resulting model has rotational symmetry in d-dimensional euclidean spacetime. This rotation symmetry is related to Lorentz symmetry by Wick rotation (section 9).

[97] Article 73054

[98] Page 75 in McGreevy (2021), and page 32 in section V.A in the preprint version of Poland *et al* (2019)

[99] This name might be a little confusing, because relevant perturbations are the ones that push the model away from that fixed point's critical manifold, at least when *critical manifold* is defined as above. Maybe the name *multicritical fixed point* comes from thinking about RG flows toward the UV instead of toward the IR.

24 References

Aizenman, 1981. "Proof of the Triviality of ϕ^4_d Field Theory and Some Mean-Field Features of Ising Models for $d > 4$" *Phys. Rev. Lett.* **47**: 1-4 and 886 (erratum)

Aizenman, 1982. "Geometric analysis of φ^4 fields and Ising models. Parts I and II" *Communications in Mathematical Physics* **86**: 1-48, https://projecteuclid.org/euclid.cmp/1103921614

Aizenman, 2021. "A geometric perspective on the scaling limits of critical Ising and φ^4_d models" https://arxiv.org/abs/2112.04248

Aizenman and Duminil-Copin, 2019. "Marginal triviality of the scaling limits of critical 4D Ising and ϕ^4_4 models" *Annals of Mathematics* **194**: 163-23, https://arxiv.org/abs/1912.07973

Ardonne and Sierra, 2010. "Chiral correlators of the Ising conformal field theory" *J. Phys. A* **43**: 505402, https://arxiv.org/abs/1008.2863

Baggio *et al*, 2018. "Decoding a three-dimensional conformal manifold" *Journal of High Energy Physics* **2018**: 62, https://link.springer.com/article/10.1007/JHEP02(2018)062

Banks, 2008. *Modern Quantum Field Theory: A Concise Introduction.* Cambridge University Press

Cappelli *et al*, 2017. "Conformal Field Theories and Renormalization Group Flows in Dimensions $d > 2$" *Activity Reports (Galileo Galilei Institute for Theoretical Physics)* **6**: 39-43, https://oajournals.fupress.net/index.php/cdg/article/download/8713/8711/

Coradeschi *et al*, 2013. "A naturally light dilaton" *JHEP* **11**: 057, https://arxiv.org/abs/1306.4601

Cresswell-Hogg and Litim, 2022. "Critical Fermions with Spontaneously Broken Scale Symmetry" https://arxiv.org/abs/2212.06815

Crewther, 2020. "Genuine Dilatons in Gauge Theories" *Universe* **6**: 96, https://arxiv.org/abs/2003.11259

Curtright *et al*, 2012. "RG flows, cycles, and c-theorem folklore" *Phys. Rev. Lett.* **108**: 131601, https://arxiv.org/abs/1111.2649

Damgaard and Thorleifsson, 1991. "Chaotic renormalization-group trajectories" *Phys. Rev. A* **44**: 2738-2741

Di Francesco *et al*, 1997. *Conformal Field Theory.* Springer

Di Francesco *et al*, 1987. "Critical Ising correlation functions in the plane and on the torus" *Nuclear Physics B* **290**: 527-581

Duminil-Copin, 2022. "100 Years of the (Critical) Ising Model on the Hypercubic Lattice" https://arxiv.org/abs/2208.00864

Fradkin, 2021. *Quantum Field Theory: an Integrated Approach.* Princeton University Press

Fradkin, 2022. "Chapter 15: The Renormalization Group" https://eduardo.physics.illinois.edu/phys583/physics583.html

Fradkin, 2022b. "Chapter 21: Conformal Field Theory" https://eduardo.physics.illinois.edu/phys583/ch21.pdf

Gerchkovitz *et al*, 2016. "Correlation Functions of Coulomb Branch Operators" https://arxiv.org/abs/1602.05971

Giuliani *et al*, 2021. "Gentle introduction to rigorous Renormalization Group: a worked fermionic example" *Journal of High Energy Physics* **2021**: 26, https://link.springer.com/article/10.1007/JHEP01(2021)026

Glazek and Wilson, 2002. "Limit cycles in quantum theories" *Phys. Rev. Lett.* **89**: 230401, https://arxiv.org/abs/hep-th/0203088

Hollowood, 2009. "6 Lectures on QFT, RG and SUSY" https://arxiv.org/abs/0909.0859v1

Jensen and O'Bannon, 2016. "A Constraint on Defect and Boundary Renormalization Group Flows" *Phys. Rev. Lett.* **116**: 091601, https://arxiv.org/abs/1509.02160

Jora, 2015. "Φ^4 theory is trivial" https://arxiv.org/abs/1503.07298

Komargodski and Schwimmer, 2011. "On Renormalization Group Flows in Four Dimensions" https://arxiv.org/abs/1107.3987

Komargodski, 2011. "The Constraints of Conformal Symmetry on RG Flows" https://arxiv.org/abs/1112.4538

Komargodski, 2012. "Comments on the Renormalization Group and Diverse Applications" https://wwwth.mpp.mpg.de/members/strings/strings2012/strings_files/program/Talks/Friday/Komargodski.pdf

Lüscher and Weisz, 1987. "Scaling laws and triviality bounds in the lattice ϕ^4 theory (I). One-component model in the symmetric phase" *Nuclear Physics B* **290**: 25-60

Lüscher, 1987. *Solution of the lattice ϕ^4 theory in 4 dimensions.* DESY 87-159, https://inspirehep.net/files/4e9ff13c445694dcbd28b9b1936e9761

McGreevy, 2019. "Physics 215C: Particles and Fields" https://mcgreevy.physics.ucsd.edu/s19/2019-215C-lectures.pdf

McGreevy, 2021. "Where do quantum field theories come from?" https://mcgreevy.physics.ucsd.edu/s14/239a-lectures.pdf

Molignini, 2013. "Analyzing the two dimensional Ising model with conformal field theory" http://edu.itp.phys.ethz.ch/fs13/cft/SM2_Molignini.pdf

Nakayama, 2015. "Scale invariance vs conformal invariance" *Physics Reports* **569**: 1-93, https://arxiv.org/abs/1302.0884

Podolsky, 2010. "On triviality of $\lambda\phi^4$ quantum field theory in four dimensions" https://arxiv.org/abs/1003.3670

Poland *et al*, 2019. "The Conformal Bootstrap: Theory, Numerical Techniques, and Applications" *Rev. Mod. Phys.* **91**: 15002, https://arxiv.org/abs/1805.04405v3

Pufu, 2016. "The F-Theorem and F-Maximization" https://arxiv.org/abs/1608.02960

Riva and Cardy, 2005. "Scale and conformal invariance in field theory: a physical counterexample" *Phys. Lett. B* **622**: 339-342, https://arxiv.org/abs/hep-th/0504197v2

Schaich and Loinaz, 2009. "An improved lattice measurement of the critical coupling in ϕ_2^4 theory" *Phys. Rev. D* **79**: 056008, https://arxiv.org/abs/0902.0045

Sen and Tachikawa, 2017. "First-order conformal perturbation theory by marginal operators" https://arxiv.org/abs/1711.05947v2

Serone, 2018. "Notes on Quantum Field Theory" https://userswww.pd.infn.it/~feruglio/Serone.pdf

Serone *et al*, 2018. "$\lambda\phi^4$ Theory I: The Symmetric Phase Beyond NNNNNNNNLO" https://arxiv.org/abs/1805.05882

Serone *et al*, 2019. "$\lambda\phi^4$ Theory II: The Broken Phase Beyond NNNN(NNNN)LO" https://arxiv.org/abs/1901.05023

Sfondrini, 2013. "Introduction to universality and renormalization group techniques" https://arxiv.org/abs/1210.2262v3

Simmons-Duffin, 2016. "TASI Lectures on the Conformal Bootstrap" https://arxiv.org/abs/1602.07982

Skinner, 2016. "The Renormalization Group" https://www.damtp.cam.ac.uk/user/dbs26/AQFT/Wilsonchap.pdf

Wilson and Kogut, 1974. "The renormalization group and the ϵ expansion" *Physics Reports* **12**: 75-200

Wu and Car, 2019. "Determination of the Critical Manifold Tangent Space and Curvature with Monte Carlo Renormalization Group" *Phys. Rev. E* **100**: 022138, https://arxiv.org/abs/1903.08231v3

Zomolodchikov, 1986. ""Irreversibility" of the flux of the renormalization group in a 2D field theory" *JETP Lett.* **43**: 730-732, http://www.jetpletters.ru/ps/1413/article_21504.pdf

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Article **38111** (https://cphysics.org/article/38111):
"Conformal Isometries in the Embedding Space Formalism" (version 2023-11-12)

Article **63548** (https://cphysics.org/article/63548):
"Scalar Quantum Fields on a Spacetime Lattice" (version 2023-11-12)

Article **73054** (https://cphysics.org/article/73054):
"Why Do Liquids Boil?" (version 2022-02-05)

Article **79649** (https://cphysics.org/article/79649):
"Renormalization Group Flow Near the Trivial Fixed Point" (version 2023-02-11)