Tensor Fields on Smooth Manifolds

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Abstract Physics is usually expressed with the help of a coordinate system, but a coordinate system is just a way of labeling the points of spacetime (or space), and nature should not care how we label things. This article introduces the concept of a **tensor field**. Explicit equations involving tensor fields are often written in terms of coordinates, which tends to obscure the fact that a tensor field is — by definition — independent of coordinates. This article defines tensor fields without using coordinates.

Contents

1 Vocabulary and notation

The word *field* has two mostly-unrelated meanings:[1]

- It can refer to an algebraic structure like the *field* of real numbers or the *field* of complex numbers. More precisely: a commutative ring is called a *field* if it has a identity element for multiplication and if every nonzero element has an inverse.[2]

- It can refer to a *tensor field*, which is the subject of this article.

The intended meaning should usually be clear from the context.

Within a given chart (article 93875), the points of an N-dimensional smooth manifold $\mathcal{M}$ may be labelled by N-tuples of real numbers. A particular N-tuple, the coordinate representation of a particular point, will be denoted

$$\mathbf{x} = (x^1,\, x^2,\, ...,\, x^N).$$

The superscripts here are indices, not exponents. The abbreviation

$$\partial_a \equiv \frac{\partial}{\partial x^a}$$

will be used for partial derivatives. Whenever the same index appears as both a superscript and a subscript within the same term, a sum over the index is implied. For example,

$$A^a B_{ab} \text{ is an abbreviation for } \sum_a A^a B_{ab}.$$

[1] The next section uses the word both ways.

[2] Pinter (1990), chapter 17, page 172

2 Smooth manifolds: where tensor fields live

Functions can be defined on any set Ω. Let $\mathbb{R}$ be the field of real numbers. A function $f:\Omega\to\mathbb{R}$ assigns one real number to each element of the set:[3]

$$p \mapsto f(p) \qquad\qquad \text{with } p\in\Omega \text{ and } f(p)\in\mathbb{R}.$$

To define continuity or derivatives of functions, the set must be endowed with some extra structure (article 93875):

- A set equipped with enough structure for defining continuity is called a **topological manifold**.
- A set equipped with enough structure for defining derivatives is called a **smooth manifold**.[4]

Tensor fields can be defined on any smooth manifold. The simplest example of a smooth manifold is $\mathbb{R}^N$, and this is the prototype from which all N-dimensional smooth manifolds are constructed patchwise. The general concept of a smooth manifold is reviewed in article 93875, but this article will lean on the reader's established intuition about the simplest example $\mathbb{R}^N$. Each N-tuple of real numbers represents a point in $\mathbb{R}^N$, and a function

$$f:\mathbb{R}^N\to\mathbb{R} \qquad\qquad (x^1,...,x^N)\mapsto f(x^1,...,x^N)$$

is called *smooth* if its *derivatives* with respect to the variables x^a are well-defined.

The thing we normally call 3d space (or 4d spacetime) is a smooth manifold together with a special tensor field called a **metric field** (section 16). Geometry is defined using the metric field (articles 21808 and 48968). On a smooth manifold by itself, without a metric field, geometric concepts like distance and angle are undefined. The definitions of tensor fields do not rely on such concepts.[5]

[3] An element of a smooth manifold is often called a **point**.

[4] This article requires that derivatives of arbitrarily high order be defined. Alternate definitions may require only that derivatives up to some finite order be defined.

[5] Tensors and tensor fields on other kinds of spaces can also be defined. This article is specifically about tensor fields on smooth manifolds, as appropriate for applications to general relativity.

3 Scalar fields: definition

The simplest type of tensor field is a **scalar field**. Given a smooth manifold $\mathcal{M}$, a scalar field S on $\mathcal{M}$ is a smooth function from $\mathcal{M}$ to $\mathbb{R}$:

$$S : \mathcal{M} \to \mathbb{R}.$$

In other words, a scalar field assigns a single real number $S(p)$ to each point $p \in \mathcal{M}$, varying smoothly from one point to the next.

If S and S' are two scalar fields, then their sum $S + S'$ and product SS' are also scalar fields, defined by

$$(S + S')(p) \equiv S(p) + S'(p). \qquad\qquad (SS')(p) \equiv S(p)S'(p).$$

If S is a scalar field and r is a real number, then rS is another scalar field:[6]

$$(rS)(p) \equiv rS(p).$$

These operations define the **algebra of scalar fields**, which will be used in section 5 to define the concept of a vector field.

A scalar field is a coordinate-independent entity, even on the manifold $\mathbb{R}^N$. Any given patch of the same manifold may be covered by $\mathbb{R}^N$ in many different ways (corresponding to many different coordinate systems), and all of them are equally legitimate descriptions of the manifold. A scalar field assigns a real number to each point of the manifold, regardless of how that part of the manifold is covered by $\mathbb{R}^N$.

[6] The right-hand side is the ordinary product of the real numbers r and $S(p)$.

4 Scalar fields: coordinate representation

Suppose that $\mathcal{M}$ is an N-dimensional smooth manifold. Within a given chart where each point p is represented by its coordinates $(x_1, ..., x_N)$, a scalar field is a smooth function $S(x_1, ..., x_N)$ of the N coordinates. However, a scalar field is a coordinate-*in*dependent entity: the real number that it assigns to a given point $p \in \mathcal{M}$ does not depend on which coordinate system we use to specify the point p.

To explain this in more detail, consider a chart (U, σ), where U is an open subset $U \subset \mathcal{M}$ and $\sigma : U \to \mathbb{R}^N$ is a homeomorphism. The chart (U, σ) defines a coordinate system for that part of $\mathcal{M}$. Given a scalar field $S : \mathcal{M} \to \mathbb{R}$ whose value at $p \in \mathcal{M}$ is $S(p)$, its coordinate representation is the function $S(\mathbf{x})$ defined by[7]

$$S(\mathbf{x})\Big|_{\mathbf{x}=\sigma(p)} \equiv S(p).$$

If we change the coordinate system by replacing σ with $\tilde{\sigma}$, then the same scalar field has the new coordinate representation

$$\tilde{S}(\mathbf{x})\Big|_{\mathbf{x}=\tilde{\sigma}(p)} \equiv S(p).$$

The relationship between the two representations is

$$\tilde{S}(\mathbf{x})\Big|_{\mathbf{x}=\tilde{\sigma}(p)} = S(\mathbf{x})\Big|_{\mathbf{x}=\sigma(p)}, \tag{1}$$

which can also be written

$$\tilde{S}(\tilde{\mathbf{x}}) \equiv S\big(\mathbf{x}(\tilde{\mathbf{x}})\big),$$

where the N-tuple of smooth functions $\mathbf{x}(\tilde{\mathbf{x}})$ converts the coordinates $\tilde{\mathbf{x}}$ of p in one system to the coordinates $\mathbf{x}$ of the same point p in the other system. These two functions (the left- and right-hand sides of (1)) are two different representations the same scalar field: they both assign the same real number to any given point of $\mathcal{M}$.

[7] To help keep the notation light, I'm using the same symbol S for two different functions: one whose input is a point $p \in \mathcal{M}$, and one whose input is an N-tuple of real numbers $\mathbf{x} \in \mathbb{R}^N$.

5 Vector fields: definition

A vector field is another type of tensor field. Let $\mathcal{S}$ denote the set of scalar fields on $\mathcal{M}$. A **vector field** V is a map from $\mathcal{S}$ to itself,

$$V : \mathcal{S} \to \mathcal{S}, \tag{2}$$

that satisfies these conditions:

$$V(S + S') = V(S) + V(S') \qquad\qquad V(rS) = r\,V(S) \tag{3}$$

and

$$V(SS') = S\,V(S') + S'\,V(S) \tag{4}$$

for all scalar fields $S, S' \in \mathcal{S}$ and all real numbers r. A map satisfying the first two conditionsn (3) is called **linear**. A map satisfying the last condition (4) is called a **derivation**. The conditions (3) and (4) are well-defined because the set of scalar fields is an algebra. Altogether, a vector field V is a linear derivation on the algebra of scalar fields.[8]

If V and V' are two vector fields, and if r is a real number, then $V + V'$ and rV are also vector fields. They are defined by

$$(V + V')(S) \equiv V(S) + V'(S) \qquad\qquad (rV)(S) \equiv r\,V(S).$$

With these definitions, the set of vector fields becomes a **vector space**.

The definition of a vector field (or any other type of tensor field) relies only on the smooth structure, not on geometric concepts like distances or angles. Geometric notions, like the norm of a vector or the angle between two vectors, are not defined on a smooth manifold. To define such concepts, we would need to specify a metric field.

[8]A more traditional coordinate-free definition of vector fields starts by defining the "tangent space" at each point of the manifold, uses that to define a vector at a point, and then defines a vector field to be a smooth choice of one vector per point. The approach used here is simpler, and it is equivalent (Lee (2013), chapter 3 and proposition 8.15 on page 181; also Isham (1999), section 2.3.5).

6 Vector fields: coordinate representation

A vector field has a direction[9] at each point of the manifold $\mathcal{M}$ where it is not zero. To see how definition shown above encodes directional information, choose a coordinate system and consider the vector field V defined by

$$(VS)(\mathbf{x}) \equiv V^a(\mathbf{x})\frac{\partial}{\partial x^a}S(\mathbf{x})$$

for all scalar fields S. The coefficients $V^a(\mathbf{x})$ are smooth functions of $\mathbf{x}$. More concisely:

$$V \equiv V^a(\mathbf{x})\frac{\partial}{\partial x^a} \tag{5}$$

The N coefficients $V^a(\mathbf{x})$ are the **components** of the vector field in this coordinate system. The field V defined by (5) clearly satisfies the conditions (3) and (4). Every vector field can be expressed this way, as a *derivative* acting on scalar fields.[10] At each point $\mathbf{x}$ for which the coefficients $V^a(\mathbf{x})$ are not all zero, the differential operator (5) defines a direction. To see how, consider a smooth curve on the manifold, described by an N-tuple of functions $\mathbf{x}(\lambda)$, specifying the coordinates of a point for each value of λ. Any scalar field $S(\mathbf{x})$ defines a function $S(\mathbf{x}(\lambda))$ along the curve. The derivative of this function with respect to λ is

$$\frac{d}{d\lambda}S\Big(\mathbf{x}(\lambda)\Big) = \left[\frac{d}{d\lambda}x^a(\lambda)\right]\frac{\partial}{\partial x^a}S(\mathbf{x})\bigg|_{\mathbf{x}=\mathbf{x}(\lambda)}.$$

At any given point along the curve, the relative magnitudes of the coefficients in square brackets encode the direction in which the curve passes through that point. Similarly, for any vector field, at any point where the coefficients $V^a(\mathbf{x})$ are not all zero, their relative magnitudes encode the direction along which the derivative is being taken. In this way, a vector field encodes a direction at each point where it is nonzero.

[9] To define the *magnitude* of a vector field requires additional structure beyond what a smooth manifold provides, such as a metric structure (defined by a metric field).

[10] Equation (8.2) in Lee (2013), and section 15.1 in Berger (2003)

7 Vector fields: coordinate transformation

A vector field is a coordinate-independent entity. It is not just a collection of components. In practice, however, calculations with vector fields are typically done using their components in a given coordinate representation.

When changing coordinate systems, the components of a vector field don't transform like scalar fields. To see how they do transform, suppose that the same point p is represented by one N-tuple $\mathbf{x} = (x_1, ..., x_N)$ in one coordinate system and by another N-tuple $\tilde{\mathbf{x}} = (\tilde{x}_1, ..., \tilde{x}_N)$ in another coordinate system. Let $V^a(\mathbf{x})$ and $\tilde{V}^a(\tilde{\mathbf{x}})$ be the components of the same vector field in the two coordinate systems. The vector field itself, the left-hand side of equation (5), does not depend on which coordinate system we use. Therefore, equation (5) implies

$$\tilde{V}^a(\tilde{\mathbf{x}})\tilde{\partial}_a = V^a\big(\mathbf{x}(\tilde{\mathbf{x}})\big)\partial_a \tag{6}$$

$$\tilde{\partial}_a \equiv \frac{\partial}{\partial \tilde{x}^a} \qquad\qquad \partial_a \equiv \frac{\partial}{\partial x^a}.$$

The left- and right-hand sides of equation (6) are the same vector field, represented in different coordinate systems.[11] Use the identity[12]

$$\partial_a = (\partial_a \tilde{x}^b)\tilde{\partial}_b, \tag{7}$$

to see that the condition (6) implies

$$\tilde{V}^a(\tilde{\mathbf{x}}) = V^b\big(\mathbf{x}(\tilde{\mathbf{x}})\big)(\partial_b \tilde{x}^a). \tag{8}$$

This says that a coordinate transformation mixes the components of a vector field with each other in a particular way. Again: the components of a vector field are not scalar fields. A vector field is a linear derivation, and this is what defines its direction at each point of the manifold.

[11] Recall the definition of a partial derivative: Each of the partial derivatives ∂_a in the first coordinate system are defined with the other coordinates in that system held fixed. Similarly, each of the partial derivatives $\tilde{\partial}_a$ in the second coordinate system are defined with the other coordinates in that system held fixed.

[12] This identity should be familiar. It makes sense because coordinate transformations are invertible, so $\tilde{\mathbf{x}}$ may be regarded as a function of $\mathbf{x}$.

8 One-forms

This section introduces another type of tensor field, called a **differential one-form**, or just a **one-form**.[13] A one-form is a linear map from the set of vector fields to the set of scalar fields. In other words, a one-form ω takes a vector field V as input and returns a scalar field $\omega(V)$ as output, subject to this condition:[14]

$$\omega(rV + r'V') = r\omega(V) + r'\omega(V')$$

for all real numbers r, r' and all vector fields V, V'.

If ω and ω' are two one-forms, and if r is a real number, then $\omega + \omega'$ and $r\omega$ are also one-forms. They are defined by

$$(\omega + \omega')(V) \equiv \omega(V) + \omega'(V) \qquad\qquad (r\omega)(V) \equiv r\,\omega(V).$$

With these definitions, the set of one-forms becomes a vector space.

Given a one-form ω and a vector V, the coordinate representation of the scalar field $S \equiv \omega(V)$ is

$$S(\mathbf{x}) = \omega_a(\mathbf{x})V^a(\mathbf{x}) \tag{9}$$

where V^a are the components of V. The functions $\omega_a(\mathbf{x})$ are the **components** of the one-form. We already know how vector fields transform (equation (8)), so we can infer from equation (9) how the components of a one-form transform:

$$\tilde{\omega}_a(\tilde{\mathbf{x}}) = \omega_b\big(\mathbf{x}(\tilde{\mathbf{x}})\big)(\tilde{\partial}_a x^b). \tag{10}$$

To deduce this, use the chain rule:

$$(\tilde{\partial}_a x^b)(\partial_b \tilde{x}^c) = \tilde{\partial}_a \tilde{x}^c = \delta_a^c.$$

[13] In section 15, after differential forms of higher degree are defined, I'll switch to writing *1-form* instead of *one-form*.

[14] This is what **linear** means.

9 A symmetry between vector fields and one-forms

We used scalar fields to define vector fields, and then we used those to define one-forms. Now that these are all defined, we have another way of thinking about vector fields. A vector field V can be regarded as a linear map from one-forms to scalar fields, namely the map defined by

$$V(\omega) \equiv \omega(V). \tag{11}$$

On the right-hand side, the vector field V is defined as before. On the left-hand side, V denotes the map defined by this new equation. This is another way of thinking about the same vector field. The symmetry of equation (11) is consistent with the symmetry of the coordinate representation (9).

10 Differentials

Section 6 explained how the coordinate representation of a vector field is naturally expressed as a differential operator. That's better than representing the vector field as just a list of components, because keeping those components packaged as a differential operator reminds us how they must transform when the coordinate system is changed. A similar way of packaging the components of a one-form exists, using the **coordinate differentials** dx^a. These are one-forms defined by

$$dx^a(\partial_b) = \delta^a_b. \tag{12}$$

They are defined on arbitrary vectors fields by linearity:

$$dx^a(V) = V^b\, dx^a(\partial_b) = V^a.$$

Now the coordinate representation of a one-form ω may be written as

$$\omega_a(\mathbf{x})\, dx^a. \tag{13}$$

Equation (12) dictates how dx^a transforms under a coordinate transformation, and then the expression (13) – together with the fact that ω is coordinate-independent – dictates that the components ω_a transform as shown in section 8.

Given any scalar field S, the **differential** of S is the one-form dS defined by

$$dS(V) \equiv V(S)$$

for all vector fields V. The coordinate representation of dS is

$$dS(\mathbf{x}) = dx^a\, \partial_a S(\mathbf{x}).$$

This looks like the usual expression for an infinitesimal variation of S, and that's not a coincidence. According to page 283 in chapter 11 of Lee (2013): "The great power of the concept of the differential comes from the fact that we can define *df* invariantly on any manifold, without resorting to vague arguments involving infinitesimals."

11 General tensor fields on smooth manifolds

The preceding sections introduced three different types of tensor field, each with its own special name. To describe general tensor fields, we can use a more systematic naming convention:

- Scalar fields are tensor fields of type $\binom{0}{0}$.
- Vector fields are tensor fields of type $\binom{1}{0}$.
- One-forms are tensor fields of type $\binom{0}{1}$.

After defining the special cases $\binom{0}{0}$ and $\binom{1}{0}$ as before, we can define tensor field of any other type $\binom{k}{m}$ for all non-negative integers k, m. A **tensor field** of type $\binom{k}{m}$ is a map that takes k one-forms and m vector fields as input, returns a single scalar field as output, and is linear in each of its inputs. The notation $\binom{k}{m}$ indicates the numbers of superscripts and subscripts in a coordinate representation of the tensor.[15]

[15] Lee (1997) uses the opposite convention: Lee's $\binom{m}{k}$ is the same as my $\binom{k}{m}$. The notation (k, m) is also used for a tensor field whose inputs are k one-forms and m vector fields.

12 Examples

A tensor field T of type $\binom{0}{2}$ takes any two vector fields V, U as input as input, returns a single scalar field $T(V, U)$ as output, and is linear in both of its inputs. The coordinate representation of the scalar field $T(V, U)$ is

$$T_{ab}(\mathbf{x})V^a(\mathbf{x})U^b(\mathbf{x}), \tag{14}$$

where $T_{ab}(\mathbf{x})$ are the **components** of T. The coordinate representation of T has two subscripts, as indicated by the notation $\binom{0}{2}$. The coordinate representation of T itself is[16]

$$T_{ab}(\mathbf{x})dx^a \otimes dx^b, \tag{15}$$

where the dx's are the basis one-forms that were defined in section 8. The **tensor product** symbol $\otimes$ acts as a separator between two positions into which separate scalar fields may be inserted. This notation reflects the fact that the two differentials dx^a and dx^b are meant to accept two separate inputs (two vector fields).

A tensor field T of type $\binom{2}{0}$ takes any two one-forms ω, ω' as input as input, returns a single scalar field $T(\omega, \omega')$ as output, and is linear in both of its inputs. The coordinate representation of the scalar field $T(\omega, \omega')$ is

$$T^{ab}(\mathbf{x})\omega_a(\mathbf{x})\omega'_b(\mathbf{x}),$$

with components $T^{ab}(\mathbf{x})$.

A tensor field T of type $\binom{1}{2}$ takes one one-form ω and two vector fields V, U as input and returns a single scalar field $T(\omega, V, U)$ as output. The coordinate representation of the scalar field $T(\omega, V, U)$ is

$$T^a{}_{bc}(\mathbf{x})\omega_a(\mathbf{x})V^b(\mathbf{x})U^c(\mathbf{x}),$$

with components $T^a{}_{bc}(\mathbf{x})$.

[16] In (15), the dx^as are one-forms, not components of one-forms. In (14), V^a and U^b are components of vector fields, not vector fields.

13 A special tensor field of type $\binom{1}{1}$

One especially important tensor field of type $\binom{1}{1}$ was used in section 10, namely the one that takes a 1-form ω and a vector field V as input and returns the scalar field $\omega(V)$ as output. The components of this special $\binom{1}{1}$ tensor field are denoted δ_a^b and are numerically given in any coordinate system by

$$\delta_a^b = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$$

so that the coordinate representation of $\omega(V)$ is

$$\omega_b V^a \, \delta_a^b = \omega_a V^a.$$

Unlike most tensor fields, this one has the same components in any coordinate system, thanks to the identity

$$(\tilde{\partial}_d x^b)(\partial_a \tilde{x}^c)\delta_c^d = \partial_a x^b = \delta_a^b.$$

14 Tensor fields with special symmetries

A tensor field of type $\binom{k}{0}$ or $\binom{0}{m}$ is called **symmetric** if it is invariant under arbitrary permutations of its inputs. Similarly, a tensor field of type $\binom{k}{0}$ or $\binom{0}{m}$ is called **antisymmetric** if it changes sign (but not magnitude) whenever two of its inputs are exchanged.

Examples: If T is a symmetric tensor field of type $\binom{0}{2}$, then

$$T(V, V') = T(V', V).$$

If T is an antisymmetric tensor field of type $\binom{0}{2}$, then

$$T(V, V') = -T(V', V).$$

15 Differential forms

An antisymmetric tensor field of type $\binom{0}{m}$ is also called an m**-form**. In the special case $m = 1$, the antisymmetry condition is trivially satisfied (because it has only one input, so there is nothing to exchange), so every tensor field of type $\binom{0}{1}$ can also be called a 1-form, as in section 8.

An anti-symmetric tensor field of type $\binom{0}{2}$ is called a **2-form**. The electromagnetic field is an important example of a 2-form.

More generally, a completely anti-symmetric tensor field of type $\binom{0}{m}$ is called an m**-form**. Completely anti-symmetric means that the sign of

$$T(V, V', V'', ...)$$

changes whenever two of its arguments are exchanged with each other. This includes 1-forms as a special case, because with only one argument, the anti-symmetry condition is trivially satisfied. m-forms are collectively called **differential forms**.

I won't try to give a proper introduction to differential forms here,[17] but I'll mention a few things to motivate further study. Differential forms are special because

- 1-forms can be integrated over curves,
- 2-forms can be integrated over surfaces,
- 3-forms can be integrated over volumes,

and so on. These integrals are coordinate-independent and independent of how the submanifold (curve, surface, and so on) is parameterized. The **exterior derivative** of an m-form ω is a $(m+1)$-form $d\omega$ (this generalizes the differential of a scalar field), and **Stokes's theorem** relates the integral of $d\omega$ over a manifold $\mathcal{M}$ to the integral of ω over the manifold's boundary $\partial\mathcal{M}$. The **exterior product** (or **wedge product**) of an m-form and an m'-form is an $(m+m')$-form. This is all part of a subject called **exterior calculus**.

[17] Chapter 16 in Lee (2013) gives a proper introduction.

16 Metric fields

A **metric field** g is a symmetric tensor field of type $\binom{0}{2}$ that is also **non-degenerate**: for every point p in the manifold, if $g(V, U)$ is zero at p for all vector fields V, then the vector field U is zero at p.

Given a metric field g and two vector fields V, U, the coordinate representation of the scalar field $g(V, U)$ is

$$g_{ab}(\mathbf{x})V^a(\mathbf{x})U^b(\mathbf{x})$$

where V^a and U^a are the compoennts of V and U. The functions $g_{ab}(\mathbf{x})$ are the **components** of the metric field. The coordinate representation of the metric field itself is

$$g_{ab}(\mathbf{x})dx^a \otimes dx^b, \tag{16}$$

as in section 12.

17 The signature of a metric field

Let g be a metric field defined on an N-dimensional manifold. At any given point $\mathbf{x}$, the components $g_{ab}(\mathbf{x})$ of the metric field may be regarded as the components of an $N \times N$ matrix. The requirement that a metric field be symmetric and non-degenerate implies that this matrix has N non-zero eigenvalues. Let p be the number of positive eigenvalues, and let n be the number of negative eigenvalues. The pair (p, n) is called the **signature** of the given metric field.

The signature is the same at all points of the manifold, and it is independent of the coordinate system. It is the same at all points of the manifold because the number of positive (or negative) eigenvalues cannot depend on $\mathbf{x}$. If it did, then at least one eigenvalue would be zero for some value of $\mathbf{x}$, but this cannot happen because the definition requires that the matrix of components always has N non-zero eigenvalues. The signature also cannot depend on which coordinate system we use, for the same reason that a similarity transformation of a matrix cannot change its eigenvalues.

Two important special cases have special names:

- The signature is called **euclidean** if all eigenvalues have the same sign. The geometry of 3d space (flat or curved) is defined using a metric field with euclidean signature (article 21808).

- The signature is called **lorentzian** if exactly one eigenvalue has opposite sign compared to the others. The geometry of spacetime (flat or curved) is defined using a metric field with lorentzian signature (article 48968).

18 Using a metric field to define duals

Given a vector field V and a 1-form ω, we can construct a scalar field $\omega(V)$, whose representation in a coordinate system $\mathbf{x}$ is

$$\omega_a(\mathbf{x})V^a(\mathbf{x}).$$

For the rest of this section, the argument $\mathbf{x}$ will be omitted. Then the preceding equation is abbreviated

$$\omega_a V^a.$$

In contrast, we cannot construct a scalar field using *only* a vector field V or using *only* a 1-form ω. When a metric field is available, we have more options. Given a metric field g and a vector field V, we can construct a scalar field as $g(V,V)$.

A metric field may be used to establish a correspondence between vector fields and 1-forms. Given a metric field g and a vector field V, feeding V into one of g's two inputs, leaving the other input free, gives a 1-form $\overline{V}$ called the **dual** of V. In coordinate-free terms, $\overline{V}$ is defined by the condition[18]

$$\boxed{\overline{V}(U) = g(V,U) \qquad\qquad \text{for all vector fields } U.} \tag{17}$$

If V^a denotes the components of $V = V^a\partial_a$ in some coordinate system, then the components of $\overline{V}$ are denoted V_a and are related to V^a by

$$V_a(\mathbf{x}) = g_{ab}(\mathbf{x})V^b(\mathbf{x}).$$

More concisely,

$$V_a = g_{ab}V^b. \tag{18}$$

The **dual** (or **inverse**) of the metric field itself may also be defined. It is a tensor field $\overline{g}$ of type $\binom{2}{0}$ defined by the condition

$$\boxed{\overline{g}(\overline{V},\omega) = V(\omega) \qquad\qquad \text{holds for all 1-forms } \omega.} \tag{19}$$

[18]The nongeneracy of the metric ensures that distinct vector fields have distinct duals.

Like g, the tensor $\overline{g}$ is symmetric. To prove this, use (19), then (11), then (19) again to get this sequence of equations:

$$\overline{g}(\overline{V},\overline{U}) = V(\overline{U}) = \overline{U}(V) = g(U,V).$$

The symmetry of the right-hand side implies the symmetry of the left-hand side.

The components of $\overline{g}$ are denoted g^{ab}. We can use the coordinate-free definition (19), to determine the relationship between the components of $\overline{g}$ and the components of g. Start with

$$\overline{g}(\overline{V},\omega) = g^{ab}V_a\omega_b = g^{ab}g_{ac}V^c\omega_b,$$

where equation (18) was used in the last step, and g^{ab} denotes the not-yet-determined components of $\overline{g}$. To determine them, use the definition (19) to get

$$g^{ab}g_{ac}V^c\omega_b = V^b\omega_b.$$

This must hold for all vector fields V and all 1-forms ω, so it implies

$$g^{ab}(\mathbf{x})g_{ac}(\mathbf{x}) = \delta^a_c.$$

Together with the symmetry $g^{ab} = g^{ba}$ that was derived in the preceding paragraph, this explains why $\overline{g}$ is called the **inverse** metric field: its components are those of the inverse of the matrix with components g_{ab}.

Now that the dual of the metric is defined, the dual of a 1-form may be defined by analogy with (17):

$$\overline{\omega}(\nu) = \overline{g}(\omega,\nu) \qquad \text{for all 1-forms } \nu. \tag{20}$$

The name *dual* is justified by these relationships:

$$\overline{\overline{V}} = V \qquad\qquad \overline{\overline{\omega}} = \omega.$$

The proof of the first relationship simply uses the preceding definitions in succession (and the proof of the second relationship is similar):

$$\overline{\overline{V}}(\omega) \equiv \overline{g}(\overline{V}, \omega) \equiv V(\omega)$$

for all 1-forms ω. The relationships

$$\overline{\omega}(\overline{V}) = \omega(V) \qquad\qquad \overline{\omega}(V) = \omega(\overline{V})$$

also hold. The proofs are straightforward applications of the definitions, with some help from (11) and symmetry.

In coordinates: given a 1-form ω with components ω_a, we can feed ω into one of the two inputs of the *inverse* metric tensor g^{-1} to get a vector field called the **dual** of ω. The components ω^a of the dual are

$$\omega^a = g^{ab}\omega_b,$$

which is the counterpart of equation (18).

In physics, we usually work with the components of tensor fields in a given coordinate system instead of working directly with their coordinate-free definitions, but knowing that tensor fields *have* coordinate-free definitions is still important. General relativity makes more sense with this perspective.

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