The Free Scalar Quantum Field: Unitary Spacetime Symmetries

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Abstract In a model whose equations of motion are linear, we can solve them explicitly: we can express the time-dependent field operators in terms of a set of time-independent operators (creation and annihilation operators), analogous to expressing a solution of a classical equation of motion in terms of data on an initial spacelike hypersurface. This article shows how those time-independent operators are affected by the unitary transformations that implement some spacetime symmetries in relativistic models of a single scalar field. The relationship of those symmetries to the stress-energy tensor is also explained. Special attention is given to Lorentz boosts and (in the massless case) scale transformations.

Contents

1 Introduction

Let $\Omega(R)$ be the set of observables associated with a region R of spacetime. In this article, a unitary[1] transformation U is called a **symmetry** if[2]

$$U^{-1}\Omega(R)U = \Omega(R_U) \tag{1}$$

for some diffeomorphism[3] of spacetime whose effect on each region is $R \to R_U$.

According to the **time-slice principle** (article 22871), all local observables should be expressible in terms of (the algebra generated by) those localized in a neighborhood of any one time. The equations of motion for the field operators can be viewed as an implicit expression of the time-slice principle.

In some models, we have the ability to solve the equations of motion explicitly, expressing all observables in terms of a fixed set of time-independent operators. In particular, in the model of a free scalar quantum field, the time-dependent field operators can all be expressed in terms of a fixed set of time-independent creation/annihilation operators – a minimum set of linearly independent operators that generate the whole algebra. This makes the time-slice property manifest. This article shows how spacetime symmetries (symmetries for which $R_U \neq R$) can be implemented as transformations of the time-independent creation/annihilation operators. The relationship between this description of symmetries and the more generally-useful expressions for the generators of those symmetries in terms of the stress-energy tensor will also be explained.

Sections 9-22 describe some examples, after some preliminaries in sections 2-8.

[1] A transformation is **unitary** if it preserves all inner products among all state-vectors.

[2] More generally, any *-automorphism σ of the algebra Ω of observables is called a *symmetry* if it satisfies $\sigma\Omega(R) = \Omega(R_U)$. If Ω includes all bounded operators on a Hilbert space, then every *-automorphism of Ω can be written as $\sigma\Omega = U^{-1}\Omega U$ for some unitary U (Arveson (1976), corollary 3 of theorem 1.4.4 in section 1.4), so in that case the two definitions of *symmetry* are equivalent.

[3] Article 93875

2 The family of models

The models in this article involve a single scalar field $\phi(t,\mathbf{x})$, where t is the time coordinate and $\mathbf{x}$ is the list of coordinates of a point in D-dimensional space:

$$\mathbf{x} = (x^1, ..., x^D)$$

The superscripts are indices, not exponents. The field's time-dependence is governed by the equation of motion[4]

$$\ddot{\phi}(t,\mathbf{x}) - \nabla^2\phi(t,\mathbf{x}) + V'\big(\phi(t,\mathbf{x})\big) = 0, \tag{2}$$

and its algebraic properties are governed by the equal-time commutation relations[5]

$$\begin{gathered}\big[\phi(t,\mathbf{x}),\phi(t,\mathbf{y})\big] = 0 \qquad\qquad \big[\dot{\phi}(t,\mathbf{x}),\dot{\phi}(t,\mathbf{y})\big] = 0 \\ \big[\phi(t,\mathbf{x}),\dot{\phi}(t,\mathbf{y})\big] = i\delta(\mathbf{x}-\mathbf{y}).\end{gathered} \tag{3}$$

For the purpose of defining the model of quantum fields without any mathematical ambiguity, space can be treated as a discrete lattice of finite size (article 52890),[6] but then symmetries like Lorentz symmetry can only hold approximately and only in the context of a restricted set of states, namely states with sufficiently low momentum compared to the lattice scale (which can be as fine as we want it to be). Most of the notation and calculations used in this article treat that approximation as though it were exact.[7]

[4] Each overhead dot denotes a derivative with respect to the time coordinate t, and ∇ is the gradient with respect to the spatial coordinates $\mathbf{x}$. The function V' is the derivative of V with respect to its argument. This article uses natural units, with $\hbar = c = 1$.

[5] $[A,B] \equiv AB - BA$

[6] That's a deficiency of this family of models, not a feature of nature itself. In some cases, like for the free scalar field, the model can be defined directly in continuous space (article 44563).

[7] If a model is not meant to be a Theory of Everything, and especially if it's only meant to be a toy model (like the models in this article), then we shouldn't be too bothered by imperfections like this. The fact that the models in this article exclude important classes of natural phenomena (like electromagnetism and gravity) is a far more relevant deficiency than the fact that they may show slight violations of Lorentz symmetry at ridiculously fine resolutions.

3 The free scalar field

For the *free* scalar field ($V' \propto \phi$), we can write down the general solution to the equation of motion in either of the forms

$$\begin{aligned} \phi(t,\mathbf{x}) &= \int \frac{d^Dp}{(2\pi)^D}\;\frac{a(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}-i\omega(\mathbf{p})t}}{\sqrt{2\omega(\mathbf{p})}} + \text{adjoint} \\ &= \int \frac{d^Dp}{(2\pi)^D}\;\frac{\tilde{a}(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}-i\omega(\mathbf{p})t}}{2\omega(\mathbf{p})} + \text{adjoint} \end{aligned} \tag{4}$$

with

$$\omega(\mathbf{p}) \equiv \sqrt{\mathbf{p}^2+m^2} \tag{5}$$

$$\tilde{a}(\mathbf{p}) \equiv a(\mathbf{p})\,\sqrt{2\omega(\mathbf{p})}. \tag{6}$$

The coefficients $a(\mathbf{p})$ (or $\tilde{a}(\mathbf{p})$) are linearly independent operators on a Hilbert space, called annihilation operators. Their adjoints $a^\dagger(\mathbf{p})$ (or $\tilde{a}^\dagger(\mathbf{p})$) are called creation operators. The normalization of $a(\mathbf{p})$ gives a more intuitive commutation relation (equation (7), below), but the normalization of $\tilde{a}(\mathbf{p})$ is more convenient for studying Lorentz symmetry (section 16).

The equal-time commutation relations (3) imply

$$[a(\mathbf{p}), a(\mathbf{p}')] = 0 \qquad \big[a(\mathbf{p}),\, a^\dagger(\mathbf{p}')\big] = (2\pi)^D\delta^D(\mathbf{p}'-\mathbf{p}) \tag{7}$$

$$[\tilde{a}(\mathbf{p}), \tilde{a}(\mathbf{p}')] = 0 \qquad \big[\tilde{a}(\mathbf{p}),\, \tilde{a}^\dagger(\mathbf{p}')\big] = (2\pi)^D\delta^D(\mathbf{p}'-\mathbf{p})\,2\omega(\mathbf{p}). \tag{8}$$

The usual Hilbert-space representation is generated by starting with a state-vector $|0\rangle$ that satisfies

$$a(\mathbf{p})|0\rangle = 0 \tag{9}$$

for all $\mathbf{p}$. Other state-vectors are constructed by acting on $|0\rangle$ with the adjoints of the operators $a(\mathbf{p})$.

4 Symmetries in terms of generators

If an operator G is self-adjoint, then the operator $U(\theta) = e^{i\theta G}$ is unitary for every real number θ. The operator G is said to **generate** the group of unitary operators $U(\theta)$.

Suppose that G generates a group of symmetries whose effect on the field operators is

$$U^{-1}(\theta)\phi(t, \mathbf{x})U(\theta) = \phi(t_\theta, \mathbf{x}_\theta), \tag{10}$$

for some diffeomorphism $(t, \mathbf{x}) \to (t_\theta, \mathbf{x}_\theta)$. Taking the derivative of both sides with respect to θ gives

$$i\big[\phi(t_\theta, \mathbf{x}_\theta),\, G\big] = \frac{d}{d\theta}\phi(t_\theta, \mathbf{x}_\theta). \tag{11}$$

Given a generator G, evaluating the left-hand side of (11) is usually easier than evaluating the left-hand side of (10). In cases where the symmetry is only approximate (as is Lorentz symmetry when the model is defined by treating space as a lattice), the approximation is easier to quantify in equation (11) than it is in equation (10). For these reasons, when considering a generic model of the form described in section 2, this article uses the differential version (11). The full unitary version (10) will be used only for a *free* scalar field, by exploiting the exact solution (4).

5 Stress-energy tensor

The generators of the symmetries studied in this article can all be expressed in terms of the components T^{ab} of the stress-energy tensor. In quantum field theory (QFT), these are self-adjoint operators on a Hilbert space. Most operators don't commute with each other ($AB \neq BA$), so we need to keep track of the order in which they are multiplied. For the family of models defined in section 2, the components of the stress-energy tensor are[8]

$$T^{ab}(t,\mathbf{x}) = \frac{(\partial^a\phi)(\partial^b\phi) + (\partial^b\phi)(\partial^a\phi)}{2} - \eta^{ab}\left(\frac{\eta^{cd}(\partial_c\phi)(\partial_d\phi)}{2} - V(\phi)\right) \tag{12}$$

where η^{ab} are the components of the (inverse) Minkowski metric,[9] and $\partial^a \equiv \eta^{ab}\partial_b$ with $\partial_b \equiv \partial/\partial x^b$. Equation (12) looks just like the stress-energy tensor for a classical scalar field,[10] after symmetrizing the operator products – writing $(AB + BA)/2$ instead of just AB – to make it manifestly self-adjoint. The main new subtlety in QFT is that the product of two field operators at the same point, like the product

$$\big(\partial^a\phi(t,\mathbf{x})\big)\big(\partial^b\phi(t,\mathbf{x})\big),$$

would be undefined in continuous space. This article avoids that issue by assuming that the model has been defined by treating space as a lattice. The lattice is fine enough that deviations from exact Lorentz symmetry are negligible (footnote 7).

For future reference, the components of T^{ab} include[8]

$$\begin{gathered} T^{00} = \frac{\dot\phi^2 + (\nabla\phi)^2}{2} + V \\ T^{0k} = T^{k0} = \frac{\dot\phi\partial^k\phi + (\partial^k\phi)\dot\phi}{2} = -\frac{\dot\phi\nabla_k\phi + (\nabla_k\phi)\dot\phi}{2}. \end{gathered} \tag{13}$$

[8] An index from the beginning of the alphabet (a, b, c) covers both space and time coordinates. An index from the middle of the alphabet (j, k) covers only the space coordinates. Index 0 indicates the time coordinate.

[9] This article uses the mostly-minus convention for the metric tensor.

[10] Article 49705

6 Stress-energy conservation

For a classical scalar field, article 49705 showed that the equation of motion implies

$$\boxed{\partial_a T^{ab} = 0.} \tag{14}$$

The derivation uses the identity

$$\partial_a(fg) = (\partial_a f)g + f\partial_a g. \tag{15}$$

When space is treated as a discrete lattice, that identity holds only up to terms of order ϵ, where ϵ is the lattice spacing (the distance between neighboring lattice sites).[11] Those terms will be neglected in this article. With that approximation, equation (14) still holds for a quantum scalar field, at least if $V(\phi)$ is a polynomial, but the derivation requires extra care because the field operator $\phi(t, \mathbf{x})$ doesn't commute with its time-derivative $\dot{\phi}(t, \mathbf{x})$. This section explains how to derive (14) when $V(\phi)$ is a polynomial, taking non-commutativity into account.

Starting with equation (12), straightforward evaluation of the derivative (using (15)) gives

$$\partial_a T^{ab} = -\frac{V'\partial^b\phi + (\partial^b\phi)V'}{2} + \partial^b V \tag{16}$$

after using the equation of motion $\eta^{ab}\partial_a\partial_b\phi + V' = 0$. This would clearly be zero if ϕ and $\partial^b\phi$ were mutually commuting. They are mutually commuting when b is a spatial index, but not when b is the time index. When b is the time index, equation (16) is

$$\partial_a T^{a0} = -\frac{V'\dot{\phi} + (\dot{\phi})V'}{2} + \dot{V}. \tag{17}$$

We will show that this is zero when $V = \phi^N$ for any integer $N \geq 0$. The fact that (16) is zero whenever V is a polynomial follows as an easy corollary.

[11] Article 71852

Use the abbreviation $\phi \equiv \phi(t, \mathbf{x})$. When space is treated as a lattice, equation (3) implies

$$[\phi, \dot{\phi}] = \frac{i}{\epsilon}$$

where ϵ is the lattice spacing. This can be used to derive the identity

$$[\phi^n, \dot{\phi}] = \frac{i}{\epsilon} n\phi^{n-1} \qquad\qquad \text{for } n \geq 1 \tag{18}$$

by induction on n, where ϕ^0 is defined to be the identity operator. The time-derivative of $V = \phi^N$ is

$$\dot{V} = \dot{\phi}\phi^{N-1} + \phi\dot{\phi}\phi^{N-2} + \phi^2\dot{\phi}\phi^{N-3} + \cdots + \phi^{N-1}\dot{\phi}.$$

The identity (18) can be used to move all factors of $\dot{\phi}$ to the left of all factors of ϕ, which gives

$$\dot{V} = N\dot{\phi}\phi^{N-1} + \big(1 + 2 + 3 + \cdots + (N-1)\big)\frac{i}{\epsilon}\phi^{N-2}. \tag{19}$$

The identity (18) also implies

$$V'\dot{\phi} = N\phi^{N-1}\dot{\phi} = N(N-1)\frac{i}{\epsilon}\phi^{N-2} + N\dot{\phi}\phi^{N-1}. \tag{20}$$

The results (19) and (20) show that the right-hand side of (17) is zero, as claimed.

Equation (14) combined with the symmetry of T^{ab} implies[12]

$$\boxed{\partial_a(T^{ab}x^c - T^{ac}x^b) = 0.} \tag{21}$$

Equations (14) and (21) will be used to help evaluate the commutator in (11) for various symmetries.

[12] In quantum field theory, the field satisfies its equation of motion by construction, so we don't need to say "if the field satisfies its equation of motion" like we would when describing conservation laws in classical field theory.

7 Effective time-independence of the generators

Formally, the generators of the symmetries studied in this article can be expressed in terms of the components T^{ab} of the stress-energy tensor. An example is the generators of translations in space, also called the momentum operators, which can be defined like this:

$$P^k \equiv \int d^D x \; T^{0k}(t,\mathbf{x})\bigg|_{t=0}. \tag{22}$$

This definition ensures that P^k is independent of time, because

$$\frac{d}{dt}\big(\text{anything evaluated at } t=0\big) = 0.$$

The generators should be independent of time because the definition (1) of a *symmetry* requires applying the same unitary operator U to all observables, no matter where they are in spacetime.[13]

On the other hand, the computational task of evaluating the commutator $[\phi(t,\mathbf{x}), P^k]$ would be much easier if P^k were expressed in terms of the field operators at time t, so that the equal-time commutation relations (3) can be used. With that motive in mind, consider the operators

$$P^k(t) \equiv \int d^D x \; T^{0k}(t,\mathbf{x}). \tag{23}$$

If the time-derivative of $P^k(t)$ is equal to zero, then we can use the quantities (22) and (23) interchangeably, making the calculation of commutators relatively easy.

Is the time-derivative of $P^k(t)$ equal to zero? The integrand in (23) depends on the time t, but that by itself doesn't necessarily prevent $P^k(t)$ from being independent of t. The conservation equation (14) implies that the time-derivative of (23) is

$$\dot{P}^k = \int d^D x \; \dot{T}^{0k}(t,\mathbf{x}) = -\int d^D x \; \nabla_k T^{0k}(t,\mathbf{x}). \tag{24}$$

[13]More carefully: we could allow the generators to include time-dependent terms that commute with all observables, because this wouldn't affect (1), but accounting for that possibility wouldn't add any useful insight.

The integrand on the right-hand side is a total derivative – it's called a **boundary term** because it depends on the fields only at the boundary of the domain of integration. We can't just sweep the boundary away by taking the domain of integration to be infinitely large, because then the integral (23) – which is a sum when space is treated as a lattice – would be undefined. We could invoke the model's definition on a lattice of finite site with periodic boundary conditions, in which case the right-hand side of (24) would be identically zero, but that wouldn't help in other cases, like the generators of rotations (section 11) and boosts (section 15), which rely on equation (21) instead of (14). In classical field theory, we can make the result zero by requiring that the field is zero outside a smaller region inside the domain of integration,[14] but that wouldn't make sense in QFT, where the fields are operators. An analog of that requirement in QFT might be to only consider states for which the field operators act like zero outside the smaller region, but that would require some tricky qualification, because the Reeh-Schlieder theorem says that physically reasonable states cannot be exactly annihilated by any strictly localized operator.

For all practical purposes, this dilemma has an easy solution. The most straightforward way to define the integrals in equations (22) and (23) is to treat space as a lattice of finite size. The size – the domain of "integration" – can be arbitrarily large, much larger than the observable universe. In practice, we only need to consider observables inside a much smaller spatial region S, like merely the size of the observable universe. Equations (3) say that field operators in distant locations (at the same time) commute with each other, so equation (24) implies that $\dot{P}_k$ commutes with $\phi(t,\mathbf{x})$ for $\mathbf{x}\in S$. I'll express this by saying that the operator $P^k(t)$ defined by (23) is **effectively** independent of time.

This approach allows us to use the equal-time commutation relations (3) to evaluate commutators like $[\phi(t,\mathbf{x}),P^k]$ relatively easily, while still using the strictly time-independent definition (22) for the generator of the symmetry. A similar approach will be used for all of the symmetries considered in this article.

[14] Article 49705

8 Boundary terms in the wavenumber domain

When specialized to a free scalar field, the generators of spacetime symmetries can be expressed in terms of the creation/annihilation operators $a(\mathbf{p})$ (or $\tilde{a}(\mathbf{p})$) defined in equations (4). An example is the generator of Lorentz boosts (section 16), which may be written in the manifestly self-adjoint form

$$\mathbf{K} = -\frac{1}{2}\int \frac{d^D p}{(2\pi)^D}\; a^\dagger(\mathbf{p}) i\nabla_{\mathbf{p}} a(\mathbf{p}) + \text{adjoint}.$$

This is nonzero, thanks to the factor of i. When space is treated as a lattice, the domain of integration over $\mathbf{p}$ is bounded.[15] In the continuum limit, the domain of integration over $\mathbf{p}$ is still effectively bounded as long as we only use states with limited momentum (limited spatial resolution). In that context, we can ignore the boundary term when integrating-by-parts, so that the generator can also be written as

$$\mathbf{K} = -\int \frac{d^D p}{(2\pi)^D}\; a^\dagger(\mathbf{p}) i\nabla_{\mathbf{p}} a(\mathbf{p}).$$

This version is more convenient for calculating the commutator of $\mathbf{K}$ with $a(\mathbf{p})$.

[15] Article 71852

9 Translations in space

As a first example, consider the transformation

$$\phi(t,\mathbf{x}) \to \phi(t,\,\mathbf{x}+\Delta\mathbf{x}), \tag{25}$$

which represents a translation in space. We could show that this is a symmetry by showing that a unitary operator U exists such that

$$U^{-1}\phi(t,\mathbf{x})U = \phi(t,\,\mathbf{x}+\Delta\mathbf{x}) \tag{26}$$

for all t and all $\mathbf{x}$. Such an operator could be written[16]

$$U = e^{i\mathbf{P}\cdot\Delta\mathbf{x}} \tag{27}$$

with $\mathbf{P} = (P^1, ..., P^D)$, where P^k is the generator of translations in the x^k-direction. Taking the gradient of equation (26) with respect to $\Delta\mathbf{x}$ and then setting $\Delta\mathbf{x} = \mathbf{0}$ gives

$$i\big[\phi(t,\mathbf{x}),\,\mathbf{P}\big] = \nabla_k\phi(t,\mathbf{x}).$$

According to equations (3), this condition is satisfied by the operators

$$\begin{aligned} P^k &\equiv \int d^Dx\; T^{0k}(t,\mathbf{x}) \\ &= -\int d^Dx\; \frac{\dot\phi(t,\mathbf{x})\nabla_k\phi(t,\mathbf{x}) + \text{adjoint}}{2} + \text{constant}, \end{aligned} \tag{28}$$

where T^{0k} is given by equation (13). Equation (14) implies that P^k is effectively independent of time, in the sense explained in section 7.

[16] This sign convention corresponds to convention $U(\Delta t) = \exp(-iH\,\Delta t)$ for the unitary time-translation operators (equation (37)). If the components of $\mathbf{P}$ are defined as in equation (28), then the generator of a translation in spacetime is $U(\Delta t, \Delta\mathbf{x}) = \exp(-iH\,\Delta t + i\mathbf{P}\cdot\Delta\mathbf{x})$. The opposite signs of the time and space terms in the exponent come from the Minkowski metric.

10 Translations in space: free field

For the free scalar field, inspection of equation (4) shows that for every time t, the translation (25) is equivalent to

$$a(\mathbf{p}) \to e^{i\mathbf{p}\cdot\Delta\mathbf{x}}a(\mathbf{p}). \tag{29}$$

This transformation clearly preserves the algebra (7), so it is unitary (footnote 2). To express the generator (28) in terms of $a(\mathbf{p})$, write U as in equation (27), take the gradient of both sides of

$$U^{-1}a(\mathbf{p})U = e^{i\mathbf{p}\cdot\Delta\mathbf{x}}a(\mathbf{p})$$

with respect to $\Delta\mathbf{x}$, and then set $\Delta\mathbf{x} = \mathbf{0}$ to get

$$i\big[a(\mathbf{p}),\, \mathbf{P}\big] = i\mathbf{p}\, a(\mathbf{p}).$$

This is satisfied by

$$\mathbf{P} = \int \frac{d^D p}{(2\pi)^D}\; \mathbf{p}\, a^\dagger(\mathbf{p})a(\mathbf{p}) + \text{constant}.$$

If we require the vacuum state-vector to be invariant under translations, then the constant term is zero.[17]

[17]The vacuum state $\rho(\cdots) \equiv \langle 0|\cdots|0\rangle/\langle 0|0\rangle$ is invariant no matter how the constant term is chosen, because translation symmetry is not spontaneously broken. The constant term only affects the overall phase of the vacuum state-vector $|0\rangle$.

11 Rotations

Now consider the rotation

$$\phi(t,\mathbf{x}) \to \phi(t,\,\mathbf{x}_R), \tag{30}$$

where $\mathbf{x}_R$ denotes a rotated version of $\mathbf{x}$. For a rotation in the j-k plane, the derivative of $\phi(t,\mathbf{x}_R)$ with respect to the rotation angle θ is[18]

$$\frac{d}{d\theta}\phi(t,\mathbf{x}_R) = (x^j\nabla_k - x^k\nabla_j)\phi(t,\mathbf{x}_R). \tag{31}$$

If rotations in the j-k plane are generated by L_{jk}, then

$$e^{-i\theta L_{jk}}\phi(t,\mathbf{x})e^{i\theta L_{jk}} = \phi(t,\mathbf{x}_R).$$

Taking the derivative of this with respect to θ and then setting $\theta = 0$ gives

$$i\big[\phi(t,\mathbf{x}),\, L_{jk}\big] = (x^j\nabla_k - x^k\nabla_j)\phi(t,\mathbf{x}).$$

According to equations (3), this condition is satisfied by

$$L_{jk} \equiv \int d^Dx\ \big(T^{0j}(t,\mathbf{x})x^k - T^{0k}(t,\mathbf{x})x^j\big) \tag{32}$$

$$= \int d^Dx\ \frac{\dot\phi(t,\mathbf{x})(x^j\nabla_k - x^k\nabla_j)\phi(t,\mathbf{x}) + \text{adjoint}}{2} + \text{constant}. \tag{33}$$

The second equality ignores the boundary term, which can be justified as in section 7. Equation (21) implies that L_{jk} is effectively independent of time, in the sense explained in section 7.

[18] I'm not trying to comply with any particular sign convention for the direction of a rotation.

12 Rotations: free field

For the free scalar field, for every time t, the rotation (30) is equivalent to

$$a(\mathbf{p}) \to a(\mathbf{p}_R), \tag{34}$$

where $\mathbf{p}_R$ is the inverse-rotated version of $\mathbf{p}$. This transformation preserves the algebra (7), so it is unitary (footnote 2). To confirm this, substitute (34) into (4) and then change the integration variable from $\mathbf{p}$ to $\mathbf{p}_R$. This leaves ω invariant and changes

$$\mathbf{p}\cdot\mathbf{x} \to \mathbf{p}_R\cdot\mathbf{x} = \mathbf{p}\cdot\mathbf{x}_R,$$

which gives (30).

To express the generator (33) in terms of $a(\mathbf{p})$, take the derivative of both sides of

$$e^{-i\theta L_{jk}}a(\mathbf{p})e^{i\theta L_{jk}} = a(\mathbf{p}_R)$$

with respect to θ and then set $\theta = 0$ to get

$$i\big[a(\mathbf{p}),\, L_{jk}\big] = \left(p^j\frac{\partial}{\partial p^k} - p^k\frac{\partial}{\partial p^j}\right)a(\mathbf{p}),$$

which is satisfied by

$$L_{jk} = -i\int \frac{d^Dp}{(2\pi)^D}\; a^\dagger(\mathbf{p})\left(p^j\frac{\partial}{\partial p^k} - p^k\frac{\partial}{\partial p^j}\right)a(\mathbf{p}) + \text{constant}.$$

The factor of i makes this self-adjoint, modulo a boundary term that can be ignored as explained in section 8. If we require the vacuum state-vector to be invariant under rotations, then the constant term is zero.

13 Translations in time

Now consider a translation in time:

$$\phi(t,\mathbf{x}) \to \phi(t+\Delta t,\,\mathbf{x}). \tag{35}$$

We could show that this is a symmetry by showing that a unitary operator U exists such that

$$U^{-1}\phi(t,\mathbf{x})U = \phi(t+\Delta t,\,\mathbf{x}) \tag{36}$$

for all t and all $\mathbf{x}$. Such an operator could be written

$$U = e^{-iH\,\Delta t}, \tag{37}$$

where H is the generator of translations in time (the hamiltonian). Taking the derivative of both sides of (36) with respect to Δt and then setting $\Delta t = 0$ gives

$$-i\big[\phi(t,\mathbf{x}),H\big] = \dot{\phi}(t,\mathbf{x}),$$

According to (3), this condition is satisfied by

$$H \equiv \int d^Dx\; T^{00}(t,\mathbf{x}) \tag{38}$$

$$= \int d^Dx\; \left(\frac{\dot{\phi}^2(t,\mathbf{x}) + \big(\nabla\phi(t,\mathbf{x})\big)^2}{2} + V\big(\phi(t,\mathbf{x})\big)\right). \tag{39}$$

Equation (14) implies that H is effectively independent of time, in the sense explained in section 7.

This agrees with the result derived in article 52890.

14 Translations in time: free field

For the free scalar field, for every time t, the time-translation (35) is equivalent to

$$a(\mathbf{p}) \to e^{-i\omega(\mathbf{p})\,\Delta t} a(\mathbf{p}). \tag{40}$$

This transformation clearly preserves the algebra (7), so it is unitary (footnote 2). To express the generator (39) in terms of $a(\mathbf{p})$, take the derivative of both sides of

$$U^{-1} a(\mathbf{p}) U = e^{-i\omega(\mathbf{p})\,\Delta t} a(\mathbf{p})$$

with respect to Δt to get

$$\big[a(\mathbf{p}), H\big] = \omega(\mathbf{p})\, a(\mathbf{p}),$$

which is satisfied by

$$H = \int \frac{d^D p}{(2\pi)^D}\; \omega(\mathbf{p})\, a^\dagger(\mathbf{p}) a(\mathbf{p}). \tag{41}$$

This agrees with the result derived in article 00980.

15 Lorentz boosts

Now consider a Lorentz boost

$$\phi(t, \mathbf{x}) \to \phi(t_B, \mathbf{x}_B). \tag{42}$$

To make this explicit, choose a fixed unit vector $\mathbf{u}$ and consider a boost in the $\mathbf{u}$-direction:[19]

$$t_B \equiv ct + s\mathbf{u}\cdot\mathbf{x} \qquad\qquad \mathbf{x}_B \equiv (st + c\mathbf{u}\cdot\mathbf{x})\mathbf{u} + (\mathbf{x} - \mathbf{u}\mathbf{u}\cdot\mathbf{x}) \tag{43}$$

with

$$c \equiv \cosh\theta \qquad\qquad s \equiv \sinh\theta.$$

The first and second parenthesis-pairs in equation (43) for $\mathbf{x}_B$ are the parts parallel and perpendicular to $\mathbf{u}$, respectively. Compared to the preceding cases, boosts are more interesting, because they mix the time and space coordinates with each other. As before, we could show that a boost is a symmetry by showing that a unitary operator U exists such that

$$U^{-1}\phi(t, \mathbf{x})U = \phi(t_B, \mathbf{x}_B) \tag{44}$$

for all t and all $\mathbf{x}$. Such an operator could be written

$$U = e^{-i\theta\mathbf{u}\cdot\mathbf{K}} \tag{45}$$

with generators

$$\mathbf{K} = (K^1, ..., K^D).$$

Taking the derivative of both sides of (44) with respect to θ and then setting $\theta = 0$ gives

$$-i\big[\phi(t, \mathbf{x}),\, \mathbf{u}\cdot\mathbf{K}\big] = \mathbf{u}\cdot\mathbf{x}\dot{\phi}(t, \mathbf{x}) + t\mathbf{u}\cdot\nabla\phi(t, \mathbf{x}). \tag{46}$$

[19] Notation: $\mathbf{u}\mathbf{u}\cdot\mathbf{x} \equiv (\mathbf{u}\cdot\mathbf{x})\mathbf{u}$.

Equations (3) imply[20]

$$\begin{aligned}
&[\phi(t,\mathbf{x}), T^{00}(t,\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y})\dot{\phi}(t,\mathbf{x}) \\
&[\phi(t,\mathbf{x}), T^{0k}(t,\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y})\partial^k\phi(t,\mathbf{x}),
\end{aligned} \tag{47}$$

so the condition (46) is satisfied by

$$K^k \equiv \int d^D x\ \big(T^{00}(t,\mathbf{x})x^k - T^{0k}(t,\mathbf{x})t\big). \tag{48}$$

This is analogous to a generator of rotations (equation (33)), but in a time-space plane instead of a space-space plane. Equation (21) can be used to show that K^k is effectively independent of time in the sense that was explained in section 7, even though the integrand has an explicit factor of t, and even though K^k doesn't commute with H.

[20]General analyses of commutation relations involving components of the stress-energy tensor are shown in equations (6)-(20) in Huang (2019) and in Boulware and Deser (1967).

16 Lorentz boosts: free field

For the free scalar field, for every time t, the boost (42) is equivalent to

$$\tilde{a}(\mathbf{p}) \to \tilde{a}\left(\mathbf{p}_B\right) \tag{49}$$

with $\tilde{a}(\mathbf{p})$ defined as in section 3 and with $\mathbf{p}_B$ defined by

$$\mathbf{p}_B = \big(c\mathbf{u}\cdot\mathbf{p} - s\omega(\mathbf{p})\big)\mathbf{u} + (\mathbf{p} - \mathbf{u}\mathbf{u}\cdot\mathbf{p}). \tag{50}$$

The two terms delineated by parentheses are the parts parallel and perpendicular to $\mathbf{u}$, respectively. To show that this is equivalent to (42) for a free field, substitute (49) into (4) and then change the integration variable from $\mathbf{p}$ to $\mathbf{p}_B$. The effect of this change of variable isn't obvious. Sections 17-18 derive the identities

$$\mathbf{p}_B\cdot\mathbf{x} - \omega\left(\mathbf{p}_B\right)t = \mathbf{p}\cdot\mathbf{x}_B - \omega(\mathbf{p})t_B \tag{51}$$

and

$$\frac{d^D p_B}{\omega(\mathbf{p}_B)} = \frac{d^D p}{\omega(\mathbf{p})}. \tag{52}$$

Use these in (4) to see that (49) is equivalent to the boost (42) for a free field. The identity (52) implies that the transformation (49) satisfies

$$2\omega\big(\mathbf{p}_B\big)\delta^D\big((\mathbf{p}')_B - \mathbf{p}_B\big) = 2\omega(\mathbf{p})\delta^D\big(\mathbf{p}' - \mathbf{p}\big). \tag{53}$$

This shows that the transformation (49) preserves the algebra (8), so it is unitary (footnote 2).

To express the generator (48) in terms of $a(\mathbf{p})$, define U as in (45), take the derivative of both sides of

$$U^{-1}\tilde{a}(\mathbf{p})U = \tilde{a}\left(\mathbf{p}_B\right)$$

with respect to θ, and then set $\theta = 0$ to get

$$\big[\tilde{a}(\mathbf{p}), \mathbf{u}\cdot\mathbf{K}\big] = \left(\frac{\partial\mathbf{p}_B}{\partial\theta}\right)_{\theta=0}\cdot\nabla\tilde{a}(\mathbf{p}) = -\omega(\mathbf{p})\mathbf{u}\cdot\nabla_{\mathbf{p}}\tilde{a}(\mathbf{p}) \tag{54}$$

where $\nabla_{\mathbf{p}}$ is the gradient with respect to $\mathbf{p}$. Use (8) and (53) to see that (54) is consistent with

$$\mathbf{K} = -\int \frac{d^Dp}{(2\pi)^D}\ \tilde{a}^\dagger(\mathbf{p}) i\nabla_{\mathbf{p}}\tilde{a}(\mathbf{p}) + \text{constant}, \tag{55}$$

which is self-adjoint in the context of states that allow $\mathbf{p}$-integration-by-parts to be used without generating boundary terms (section 8). If we require the vacuum state-vector to be invariant under rotations, then the constant term is zero.

To check this result, use equation (4) to get

$$\begin{aligned} \mathbf{u}\cdot\mathbf{x}\dot{\phi} + t\mathbf{u}\cdot\nabla\phi &= -\int \frac{d^Dp}{(2\pi)^D}\ \frac{\tilde{a}(\mathbf{p})\omega\mathbf{u}\cdot\nabla_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}-i\omega(\mathbf{p})t}}{2\omega(\mathbf{p})} + \text{adjoint} \\ &= -\int \frac{d^Dp}{(2\pi)^D}\ \frac{\tilde{a}(\mathbf{p})\mathbf{u}\cdot\nabla_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}-i\omega(\mathbf{p})t}}{2} + \text{adjoint} \end{aligned} \tag{56}$$

When $\mathbf{p}$-integration-by-parts can be used without generating boundary terms, this is consistent with the combination of equations (46) and (55).

17 Derivation of (51)

To derive (51), start with the definition of the function ω, which may be written

$$\big(\omega\big(\mathbf{p}_B\big)\big)^2 - \big(\mathbf{p}_B\big)^2 = m^2 \qquad\qquad \omega\big(\mathbf{p}_B\big) > 0. \tag{57}$$

Use $c^2 - s^2 = 1$ to confirm the identity

$$\big(c\omega(\mathbf{p}) - s\mathbf{u}\cdot\mathbf{p}\big)^2 - \big(c\mathbf{u}\cdot\mathbf{p} - s\omega(\mathbf{p})\big)^2 = \big(\omega(\mathbf{p})\big)^2 - (\mathbf{u}\cdot\mathbf{p})^2,$$

and use the definition of $\mathbf{p}_B$ to rewrite this as

$$\big(c\omega(\mathbf{p}) - s\mathbf{u}\cdot\mathbf{p}\big)^2 - \big(\mathbf{u}\cdot\mathbf{p}_B\big)^2 = \big(\omega(\mathbf{p})\big)^2 - (\mathbf{u}\cdot\mathbf{p})^2.$$

The part of $\mathbf{p}$ perpendicular to $\mathbf{u}$ not affected by the boost (50), so we can also write this as

$$\big(c\omega(\mathbf{p}) - s\mathbf{u}\cdot\mathbf{p}\big)^2 - \big(\mathbf{p}_B\big)^2 = \big(\omega(\mathbf{p})\big)^2 - \mathbf{p}^2 = m^2,$$

where the definition of $\omega(\mathbf{p})$ was used in the last step. Compare this to equation (57) to infer

$$\big(\omega\big(\mathbf{p}_B\big)\big)^2 = \big(c\omega(\mathbf{p}) - s\mathbf{u}\cdot\mathbf{p}\big)^2.$$

The sign of $\omega\big(\mathbf{p}_B\big)$ is fixed by requiring that it reduces smoothly to $\omega(\mathbf{p})$ as $\theta \to 0$, so the correct square root of the preceding equation is

$$\omega\big(\mathbf{p}_B\big) = c\omega(\mathbf{p}) - s\mathbf{u}\cdot\mathbf{p}, \tag{58}$$

which implies (51). This completes the derivation of equation (51).

18 Derivation of (52)

To derive (52), use the definition (50) of $\mathbf{p}_B$ to get

$$d\mathbf{p}_B = \left(c\mathbf{u}\cdot d\mathbf{p} - \frac{s}{\omega}\mathbf{p}\cdot d\mathbf{p}\right)\mathbf{u} + (d\mathbf{p} - \mathbf{u}\mathbf{u}\cdot d\mathbf{p}).$$

The part perpendicular to $\mathbf{u}$ is

$$(d\mathbf{p}_B)_\perp = d\mathbf{p}_B - \mathbf{u}\mathbf{u}\cdot d\mathbf{p}_B = d\mathbf{p} - \mathbf{u}\mathbf{u}\cdot d\mathbf{p} = (d\mathbf{p})_\perp \tag{59}$$

and, with the help of equation (58), the part parallel to $\mathbf{u}$ is

$$\begin{aligned} \mathbf{u}\cdot d\mathbf{p}_B &= c\mathbf{u}\cdot d\mathbf{p} - \frac{s}{\omega}\mathbf{p}\cdot d\mathbf{p} \\ &= \frac{\omega(\mathbf{p}_B)}{\omega(\mathbf{p})}\mathbf{u}\cdot d\mathbf{p} - \frac{s}{\omega(\mathbf{p})}\mathbf{p}\cdot(d\mathbf{p})_\perp. \end{aligned} \tag{60}$$

Equations (59) and (60) imply (52), because the last term in (60) does not contribute to $d^D p_B$. A quick way to check this assertion is to think of $d^D p_B$ as the wedge product[21] of the D components of $d\mathbf{p}_B$, which is equal to the determinant of the transformation times the wedge product of the D components of $d\mathbf{p}$.[22] Then the assertion follows easily from the antisymmetry of the wedge product.

[21] Article 81674

[22] As explained in article 81674, this can be used as the *definition* of the determinant.

19 Lorentz symmetry in the single-particle sector

For the free scalar field, states of the form

$$\int \frac{d^D p}{(2\pi)^D}\; \psi(\mathbf{p}) a^\dagger(\mathbf{p})|0\rangle$$

are single-particle states. The generators of spacetime symmetries don't mix states with different numbers of particles in this model, so we can restrict the set of observables to those that don't mix single-particle states with other states. Then we can discard the other states to get a self-contained Lorentz-symmetric model of a single particle. This model can be formulated entirely in terms of the functions $\psi(\mathbf{p})$. The inner product is

$$\langle\psi_1|\psi_2\rangle = \int d^D p\; \psi_1^*(\mathbf{p})\psi_2(\mathbf{p}), \tag{61}$$

and the hamiltonian is

$$H = \omega(\mathbf{p}).$$

Equations (6) and (49) show that the effect of a boost is

$$\psi(\mathbf{p}) \to \sqrt{\frac{\omega(\mathbf{p})}{\omega(\mathbf{p}_B)}}\, \psi(\mathbf{p}_B).$$

The inner product (61) is Lorentz-invariant because the measure automatically transforms as

$$d^D p \to \frac{\omega(\mathbf{p}_B)}{\omega(\mathbf{p})}\, d^D p_B.$$

Altogether, this is a self-contained Lorentz-symmetric model of a single quantum particle, but beware: this model doesn't have any observables that are localized entirely in a bounded region of space.[23] Local observables necessarily mix states with different numbers of particles.

[23] Article 30983

20 Massless free field: stress-energy tensor

The model with $m = 0$ has symmetries that the $m > 0$ model doesn't have. One of these is the shift symmetry described in article 37301. The shift symmetry is an **internal symmetry**: it doesn't mix operators in different regions of spacetime with each other. In the continuum ($\epsilon \to 0$) and infinite-volume ($L \to \infty$) limits, the $m = 0$ model also has **scale symmetry** (also called **dilation symmetry** or **scale invariance**), which does mix operators in different regions of spacetime with each other. Section 21 studies the scale symmetry of the massless free field.

To streamline the analysis, section 21 uses a modified stress-energy tensor. This section describes the modification and then explains why it's useful, why it's natural, and why it's harmless.

The massless free field corresponds to taking V to be a constant C (times the identity operator), so that $V' = 0$ in the equation of motion (2).[24] To streamline the analysis in section 21, a modified stress-energy tensor will be used, namely[25]

$$\tilde{T}^{ab} \equiv T^{ab} + (\eta^{ab}\partial^2 - \partial^a\partial^b)(\xi\phi^2) \tag{62}$$

where T^{ab} is the original stress-energy tensor shown in equation (12), and

$$\xi \equiv \frac{D-1}{4D}. \tag{63}$$

Regardless of the value of ξ, the original conservation equation (14) implies

$$\partial_a \tilde{T}^{ab} = 0. \tag{64}$$

To see why the value (63) is special, consider the trace of the stress-energy tensor. For $V = C$, the trace of the original stress-energy tensor is

$$\eta_{ab}T^{ab} = \frac{1-D}{2}(\partial\phi)^2 + (1+D)C.$$

[24] We won't assume $C = 0$, because then the generator of time translations (equation (39)) would assign an energy to the vacuum state that diverges in the continuum limit (article 00980).

[25] Alves (2022) summarizes various calculations related to this modification.

With the help of the equation of motion $\partial^2\phi = 0$, the trace of the modified stress-energy tensor is

$$\eta_{ab}\tilde{T}^{ab} = (1+D)C. \tag{65}$$

The important thing is that this is a constant (proportional to the identity operator), so it commutes with the field operators. The next paragraph explains why this is useful.

Section 21 will write the generator of scale transformations as

$$S \equiv \int d^Dx\ \big(\eta_{ab}\, x^a\tilde{T}^{0b}(t,\mathbf{x})\big). \tag{66}$$

The modification described above is useful because (64) implies

$$\partial_c(\eta_{ab}\, x^a\tilde{T}^{cb}) = \eta_{ab}\tilde{T}^{ab},$$

and equation (65) says that this commutes with the field operators. We can use this to show that the generator (66) is effectively independent of time, in the sense explained in section 7. Section 7 already explained why this is a useful property.

To understand why the extra term in (62) is natural, recall[26] that in classical field theory, the stress-energy tensor is naturally defined as a variation of the action with respect to the metric field. Even if we're only interested in flat spacetime, defining this variation requires at least temporarily generalizing the action to an arbitrary metric field. The generalization is not unique, so the stress-energy tensor is also not unique, not even after specializing to a flat metric field (which can only be done after computing the variation). The modified stress-energy tensor (62) is what we get when the action is taken to depend on the metric field as described near the end of article 10254.[27] That special choice is natural because it endows the model – at least the non-quantum version of the model[28] – with a large group of extra symmetries.

[26] Articles 11475, 37501, and 32191

[27] When comparing this article to that one, remember that the quantity D here is the number of spatial dimensions, not spacetime dimensions.

[28] The cited articles did not consider the quantum version.

In addition to being useful and natural, the extra term in (62) is also harmless, in the sense that replacing T^{ab} with $\tilde{T}^{ab}$ would not ruin any of the results derived in the preceding sections for other symmetries. To check this, notice that the double-time-derivatives in the extra term (the term involving ξ) cancel each other, so the extra term is a total derivative with respect to $\mathbf{x}$. This can be used to confirm that the contribution of the extra term in (62) to the generators in the preceding sections involves only the field operators at the boundary of the domain of integration, so their commutators with field operators in the region of interest is not affected (section 7).

21 Massless free field: scale symmetry, part 1

For any given scale factor $\beta > 0$, the corresponding scale transformation is

$$\phi(t, \mathbf{x}) \to \beta^{(D-1)/2}\phi(\beta t, \beta\mathbf{x}). \tag{67}$$

Section 22 will show that the overall factor $\beta^{(D-1)/2}$ is needed for the transformation to be unitary.[29] It can also be motivated by observing that the effects of the replacement (67) on the generators P^k and H of translations in space and time (equations (28) and (39), respectively) are $P^k \to P^k/\beta$ and $H \to H/\beta$, which is consistent with interpreting these operators as the total momentum and total energy observables, respectively.

We could show that the transformation (67) is a symmetry by showing that a unitary operator U exists such that

$$U^{-1}\phi(t, \mathbf{x})U = \beta^{(D-1)/2}\phi(\beta t, \beta\mathbf{x}) \tag{68}$$

for all t and all $\mathbf{x}$. Such an operator could be written

$$U = e^{-i\theta S}$$

where S is the generator and $\beta = e^{\theta}$. Taking the derivative of both sides of (68) with respect to θ and then setting $\theta = 0$ gives

$$-i\big[\phi(t, \mathbf{x}),\, S\big] = \left(\frac{D-1}{2} + t\partial_t + \mathbf{x}\cdot\nabla\right)\phi(t, \mathbf{x}). \tag{69}$$

The next paragraph shows that the operator (66) satisfies this condition, and section 21 already showed that it is effectively independent of time, in the sense explained in section 7.

[29] Article 10254 gives more insight about the factor $\beta^{(D-1)/2}$ in the context of classical field theory, by recognizing the scale transformation as a special case of a more general pattern. (When comparing this article to that one, remember that the quantity D here is the number of spatial dimensions, not spacetime dimensions.)

To show that (66) satisfies (69), use the abbreviation $\phi \equiv \phi(t, \mathbf{x})$, and use (62) to get

$$\begin{aligned}\tilde{T}^{00} &= T^{00} + \Gamma \\ \tilde{T}^{0k} &= T^{0k} - \xi\partial^k(\phi\dot{\phi} + \dot{\phi}\phi)\end{aligned}$$

where Γ doesn't involve $\dot{\phi}$, so it commutes with ϕ. Use these to see that (66) may be written

$$\begin{aligned}S &= \int d^Dx\ \left(t\tilde{T}^{00} - \sum_k x^k\tilde{T}^{0k}\right) \\ &= \int d^Dx\ \left(tT^{00} + t\Gamma - \sum_k x^k T^{0k} - \xi\mathbf{x}\cdot\nabla(\phi\dot{\phi} + \dot{\phi}\phi)\right) \\ &= \int d^Dx\ \left(tT^{00} + t\Gamma - \sum_k x^k T^{0k} - \nabla\cdot\big(\xi\mathbf{x}(\phi\dot{\phi} + \dot{\phi}\phi)\big) + D\xi(\phi\dot{\phi} + \dot{\phi}\phi)\right) \\ &= \int d^Dx\ \left(tT^{00} - \sum_k x^k T^{0k} + D\xi(\phi\dot{\phi} + \dot{\phi}\phi)\right) + \text{terms that don't matter,}\end{aligned}$$

where the "terms that don't matter" are terms that commute with ϕ at the points of interest, which are far away from the boundary of the domain of integration. With this expression for the generator S, the fact that it satisfies (69) is clear from equations (3) and (47).

22 Massless free field: scale symmetry, part 2

For the massless free scalar field, the transformation

$$a(\mathbf{p}) \to \beta^{-D/2} a(\mathbf{p}/\beta)$$

is equivalent to (67), because its effect on the field operator (4) is

$$\begin{aligned}\phi(t,\mathbf{x}) &\to \beta^{-D/2}\int \frac{d^D p}{(2\pi)^D}\,\frac{e^{-i\omega(\mathbf{p})t}e^{i\mathbf{p}\cdot\mathbf{x}}}{\sqrt{2\omega(\mathbf{p})}}a(\mathbf{p}/\beta) + \text{adjoint}\\ &= \beta^{D/2}\int \frac{d^D p}{(2\pi)^D}\,\frac{e^{-i\omega(\beta\mathbf{p})t}e^{i\beta\mathbf{p}\cdot\mathbf{x}}}{\sqrt{2\omega(\beta\mathbf{p})}}a(\mathbf{p}) + \text{adjoint}.\end{aligned} \tag{70}$$

In the massless case, $\omega(\beta\mathbf{p}) = \beta\omega(\mathbf{p})$, so (70) reduces to (67).

Use the identity

$$\beta^{-D}\delta(\mathbf{p}/\beta) = \delta(\mathbf{p})$$

to see that equation (7) implies

$$\Big[\beta^{-D/2}a(\mathbf{p}/\beta), \beta^{-D/2}a^\dagger(\mathbf{p}'/\beta)\Big] = [a(\mathbf{p}), a^\dagger(\mathbf{p}')].$$

This shows that the scale transformation (67) is unitary (footnote 2), so it is a symmetry of the massless free scalar field.

23 Massless free field: conformal symmetry

I won't prove this here, but the free scalar model is a **conformal field theory (CFT)**. A CFT is a QFT that has symmetries[30] corresponding to all conformal isometries,[31] at least those that are continuously connected to the identity transformation.

The conformal symmetry of the free scalar model isn't usually expressed in terms of explicit unitary transformations of the creation/annihilation operators, like section 22 did for scale symmetry. Swieca and Völkel (1973) is a rare exception. Conformal symmetry is more easily exploited using other formulations instead.

Well-behaved quantum field models with scale symmetry in four-dimensional spacetime tend to also have conformal symmetry, but the reason for this tendency is not yet fully understood.[32]

[30]Some of these symmetries are only defined almost everywhere in spacetime, not quite everywhere. Article 10254 described this in more detail in the context of classical field theory.

[31]Article 38111

[32]This is reviewed in Nakayama (2013), and more concisely in section 6.4 in Qualls (2015). This tendency is relatively well-understood for two-dimensional spacetime (section 6.2 in Qualls (2015)), and it seems to be absent in spacetimes of dimension greater than four (end of section 6.4 in Qualls (2015)). The first paragraph in Fitzpatrick *et al* (2013) says that no *nontrivial* CFTs are known in N-dimensional spacetime with $N \geq 7$, and the first paragraph in Gadde and Sharma (2022) suggests that such CFTs probably don't exist. The free scalar model is a *trivial* CFT.

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