Topological Operators and Higher-Form Symmetries

Randy S

Abstract This article introduces **topological operators** and the **higher-form symmetries** they generate. In d-dimensional spacetime, a topological operator nominally localized on a submanifold with q dimensions defines a symmetry that can affect some operators that are nominally localized on submanifolds with p dimensions, provided $p + q \geq d - 1$ so that the submanifolds can be linked with each other in the knot-theoretic sense. This is called a ***p*-form symmetry**. This article explains the relationship between how these symmetries are described in the canonical and path integral formulations. Examples with $p = 0$ (**zero-form symmetry**) and with $p = 1$ (**one-form symmetry**) are given, including a one-form symmetry called **center symmetry**.

Contents

1 Introduction and outline

The concept of a *higher-form symmetry* is a natural generalization of the traditional concept of symmetry in quantum field theory.[1] The concept is most natural in the path integral formulation, where operators are represented as modifications of the integrand of the path integral. In that representation, an operator is more than just a linear transformation of the Hilbert space,[2] and the extra information is important for the full definition of higher-form symmetries. This article uses both the path integral and canonical (operators-on-a-Hilbert-space) formulations and explains the relationship between their respective descriptions of higher-form symmetries.

This article has three parts:

- Sections 4-18 introduce the general concepts.
- Sections 19-30 explore an easy example of a zero-form symmetry.[3]
- Sections 31-34 review examples of one-form symmetries involving **Wilson operators** and **'t Hooft operators**.[4] Those sections cite other articles for more detail.

An appendix (section 35) addresses a technical issue about how certain operators are represented the path integral formulation.[5]

[1]Etxebarria (2022) lists this among a collection of fruitful extensions of the traditional concept of *symmetry*.

[2]Article 02242

[3]These sections assume familiarity with the material about quantum scalar fields in articles 37301, 52890, and 63548.

[4]Article 22721 gives an overview of these operators.

[5]The issue is demonstrated using a scalar field in one-dimensional spacetime, but it is present for spacetimes with any number of dimensions and also applies to the representation of topological 't Hooft operators in the Villain model of compact electrodynamics (not reviewed here).

2 Notation and conventions

The d-dimensional spacetime manifold M_{st} is **globally hyperbolic**, so it is homeomorphic to $\mathbb{R} \times M_{\text{s}}$. The first factor is **time**. The second factor M_{s} is the $(d-1)$-dimensional **spatial manifold**. For simplicity, this article assumes that M_{s} is closed, like a torus. A point in spacetime will be denoted either x or $(\mathbf{x}, t)$, using boldface for the spatial part and t for time, and coordinates are denoted x^k:

$$x = (x^0, x^1, ...x^{d-1}) \qquad \mathbf{x} = (x^1, ..., x^{d-1}) \qquad t \equiv x^0.$$

The partial derivative with respect to the kth spacetime coordinate is abbreviated

$$\partial_k \equiv \frac{\partial}{\partial x^k}.$$

More vocabulary:

- A **closed manifold** is a compact manifold without a boundary.
- The unqualified name **submanifold** will mean **properly embedded submanifold**.[6] Roughly, a submanifold is *properly embedded* if it does not intersect itself and is not missing any part of its boundary.

Given an oriented[7] manifold X, its orientation-reversed counterpart will be denoted X^{-1}.

[6] Article 44113, version 2025-11-01 or later

[7] Article 91116 defines **orientation**.

3 Two meanings of closed

In the context of topology, the word **closed** is used with two different meanings:

- A topological structure for X is defined by designating some subsets of X to serve as the *open subsets* of X. The complement of an open set is a **closed subset** of X.

- A compact manifold whose boundary is empty is called a **closed manifold**.

This article uses the word *closed* both ways. It is used with the first meaning only when it modifies the noun *subset*. Otherwise, the second meaning should understood.

4 Operators

In its most basic form, a path integral gives the result of evolution of a given initial state through a given time interval in the Schrödinger picture. An operator is described by modifying the integrand of the path integral. A modification of the integrand of the path integral is more than just an operator acting on states in the Hilbert space, though. This becomes clear when two such modifications are both included together in the path integral. Article 02242 explains this in generic terms, and it is important in the study of higher-form symmetries. In this article, $\mathcal{A}$ denotes the algebra of operators when they are regarded as nothing more than operators on the Hilbert space, and $\mathcal{M}$ denotes the set of modifications of the integrand that represent operators in $\mathcal{A}$. Elements of $\mathcal{M}$ will still be called **operators**, even though the map $\mathcal{M} \to \mathcal{A}$ is forgetful: an element of $\mathcal{M}$ is more than just an element of $\mathcal{A}$.

As in article 02242, the elements of $\mathcal{M}$ that are described by modifications of the integrand involving only integration variables in the region R will be denoted $\mathcal{M}(R)$ and will be called **localized in** R. The result of applying the forgetful map $\mathcal{M} \to \mathcal{A}$ to $\mathcal{M}(R)$ will be denoted $\mathcal{A}(R)$.

Two operators A and B that differ as elements of $\mathcal{M}$ may be equal as elements of $\mathcal{A}$, so relationships like $A = B$ or $A \neq B$ require clarification. This article writes

$$A \overset{\mathcal{M}}{=} B \qquad\qquad A \overset{\mathcal{M}}{\neq} B$$

to indicate that A and B are equal/unequal to each other as elements of $\mathcal{M}$, and

$$A \overset{\mathcal{A}}{=} B \qquad\qquad A \overset{\mathcal{A}}{\neq} B$$

to indicate that they are equal/unequal to each other as elements of $\mathcal{A}$. Then

$$A \overset{\mathcal{M}}{=} B \;\;\text{ implies }\;\; A \overset{\mathcal{A}}{=} B,$$

but

$$A \overset{\mathcal{A}}{=} B \;\;\text{ does not imply }\;\; A \overset{\mathcal{M}}{=} B.$$

5 Composition of operators

The way two operators A and B are composed depends on whether they are regarded as elements of $\mathcal{A}$ or as elements of $\mathcal{M}$:

- If two operators A and B are regarded as elements of $\mathcal{A}$, then they can be composed by applying them successively, denoted AB or BA depending on the order in which they are applied. This is the usual **algebraic product**.
- If two operators A and B are regarded as elements of $\mathcal{M}$, then they can be composed by including both modifications in the integrand of the path integral. The result of this composition will be denoted $\tau(A, B)$.

If the regions in which A and B are localized do not both intersect each others' causal pasts, then $\tau(A, B)$ reduces to the usual *time-ordered product*, which is proportional to one of the algebraic products AB or BA when regarded as an element of $\mathcal{A}$.[8]

[8] Article 02242

6 Operators on lower-dimensional submanifolds

In smooth spacetime, observables are typically not strictly localized on lower-dimensional submanifolds,[9] but having a concise language for operators that are close to being localized on lower-dimensional submanifolds will be convenient. Let X be a lower-dimensional submanifold X of spacetime, and suppose $\mathcal{O}$ is localized in some narrow tubular neighborhood[10] X_{nbhd} of X. The operator $\mathcal{O}$ will be called **localized on** X, and the pair $(X, \mathcal{O})$ will be abbreviated $\mathcal{O}(X)$. The abbreviations $\mathcal{M}(X) \equiv \mathcal{M}(X_{\text{nbhd}})$ and $\mathcal{A}(X) \equiv \mathcal{A}(X_{\text{nbhd}})$ will also be used, so:

- $\mathcal{O}(X)$ is an element of $\mathcal{M}(X)$ when it is regarded as a modification of the integrand of the path integral,
- $\mathcal{O}(X)$ is an element of $\mathcal{A}(X)$ when it is regarded as nothing more than a linear operator on the Hilbert space.

If the intersection $X_1 \cap X_2$ is empty, then the composition $\tau\big(\mathcal{O}_1(X_1), \mathcal{O}_2(X_2)\big)$ is defined and is localized on the union $X_1 \cup X_2$:[11,12]

$$\tau\big(\mathcal{O}_1(X_1), \mathcal{O}_2(X_2)\big) \in \mathcal{M}(X_1 \cup X_2).$$

This generalizes in the obvious way to the composition of any number of operators whose localizations regions don't intersect each other.

[9] Article 10690

[10] Article 53600 defines **tubular neighborhood**. Using such a neighborhood is a concession for the fact that observables typically cannot be strictly localized on lower-dimensional manifolds in continuous spacetime. *Narrow* won't be quantified, but it is meant to be small (in the directions transverse to X) compared the separations between where the operators in any composition of interest are localized.

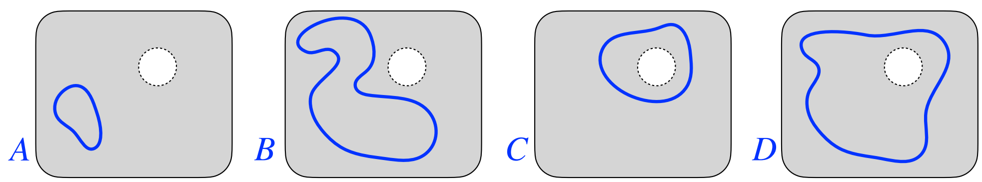
[11] Article 02242

[12] If $X_1 \cap X_2$ is not empty, then $\tau\big(\mathcal{O}_1(X_1), \mathcal{O}_2(X_2)\big)$ might be undefined, because $\mathcal{O}_1(X_1)$ and $\mathcal{O}_2(X_2)$ might prescribe mutually inconsistent modifications of the integrand of the path integral.

7 The concept of a continuous deformation

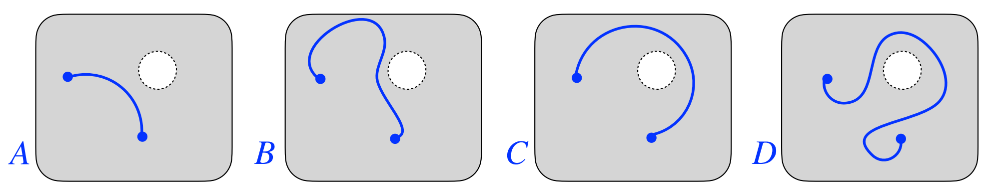
Let X be a submanifold of higher-dimensional manifold M. A **continuous deformation** of X is a continuous series of infinitesimal changes that all preserve the topology and proper-submanifoldness of X.[13] Such a series may be viewed as the image of a **homotopy** (article 61813) that is a homeomorphism for each value of the deformation parameter.

These pictures show four different 1-dimensional submanifolds (the blue curves labeled A, B, C, D) of a 2-dimensional ambient manifold that has a hole in it:



Submanifolds A and B are continuous deformations of each other, and C and D are continuous deformations of each other, but A and C are not continuous deformations of each other because one cannot be obtained from the other by a continuous series of infinitesimal changes without breaking the curve.

A continuous deformation of X **preserves the boundary** if the boundary of X remains unchanged throughout the process. These pictures show four different 1-dimensional submanifolds, all with the same boundary (same pair of endpoints):



Submanifolds A and B are boundary-preserving continuous deformations of each other, and C and D are boundary-preserving continuous deformations of each other, but A and C are not boundary-preserving continuous deformations of each other.

[13] The name *continuous deformation* is standard in the physics literature (example: Harlow and Ooguri (2021)).

8 Topological operators

Let X_0 be a submanifold of spacetime, and let $[X_0]$ be the family of submanifolds that can be obtained from X_0 by continuous deformations of X_0 that preserve its boundary. A family of operators $U(X) \in \mathcal{M}(X)$, one for each $X \in [X_0]$, is called a **topological operator** if it satisfies this condition:[14,15]

Topological invariance condition

For each $X \in [X_0]$ and each operator $\mathcal{O}(Y) \in \mathcal{M}(Y)$, the operator $\tau\big(U(X), \mathcal{O}(Y)\big) \in \mathcal{A}$ is invariant under all boundary-preserving continuous deformations of X that don't intersect Y during the deformation process.

The name **topological operator** is also used for each individual representative $U(X)$ of the family. The operators $U(X)$ in the family are not all equal to each other as elements of $\mathcal{M}$, but the invariance condition implies that they are all equal to each other as elements of $\mathcal{A}$.[16,17]

An operator can satisfy this invariance condition without being localized on the manifold's boundary ∂X_0. That fact is the foundation for a fruitful generalization the concept of symmetry. This article introduces that generalization with emphasis on the two most widely-used cases: zero-form symmetries and one-form symmetries.

[14] Sections 4-5 defined $\tau(\cdots)$ and explained what *operator* means here.

[15] This definition of *topological operator* is more general than the definition of *symmetry operator* in section 9. Sometimes the definition of *topological operator* might include the invertibility condition that section 9 imposes on symmetry operators. That condition is omitted here to accommodate the surface-localized Wilson operators $W^\bullet$ defined in article 40191, which are not invertible when the surface has a boundary and the gauged group is nonabelian.

[16] To deduce this, take $\mathcal{O}(Y)$ to be the identity operator.

[17] Bhardwaj *et al* (2024), text around equations (2.39)-(2.40)

9 Symmetry operators

Let X be an oriented submanifold of spacetime. An invertible topological operator $U(X)$ is also called a **symmetry operator**.[18,19] The examples in this article have the property

$$U\big(X^{-1}\big) = \big(U(X)\big)^{-1} \tag{1}$$

where X^{-1} is the orientation-reversed version of of X.[20]

Suppose X has a boundary, and let X_1 and X_2 be continuous deformations of X that preserve both its boundary and its orientation, and suppose that $X_1 \cup X_2^{-1}$ is itself the boundary of a higher-dimensional submanifold. Then we can define a topological operator $U(X_1 \cup X_2^{-1})$ localized on the closed manifold $X_1 \cup X_2^{-1}$ by[21,22]

$$\begin{aligned} \tau\big(U(X_1 \cup X_2^{-1}), ...\big) &\overset{\mathcal{M}}{\equiv} \tau\big(U(X_1), U(X_2^{-1}), ...\big) \\ &\overset{\mathcal{M}}{\equiv} \tau\Big(U(X_1), \big(U(X_2)\big)^{-1}, ...\Big). \end{aligned} \tag{2}$$

Given a symmetry operator $U(X)$, section 11 will define the corresponding *symmetry* in a way that works whether X has a boundary or not. Section 17 will use the relationship (2) to relate the with-boundary and without-boundary versions to each other.

---

[18]Bhardwaj *et al* (2024), definition 2.1; Luo *et al* (2024), equation (2.3)

[19]**Non-invertible symmetry** is another generalization of the concept of *symmetry* in which the topological operators implementing the symmetry are not required to be invertible (Davighi (2025), section 2; Córdova *et al* (2024), second paragraph).

[20]Section 2

[21]Section 5 defined $\tau(\cdots)$.

[22]If we were trying to make this definition precise in continuous spacetime, we would need to address exactly what happens at the shared boundary of X_1 and X_2. Most models of quantum fields in 3- or 4-dimensional spacetime have never been constructed directly in continuous spacetime, though. The examples that will be used in this article are precisely defined by treating spacetime as a lattice (article 82508), and the boundary of X in these examples does not intersect any sites, links, or (when the operator described as a modification of the integrand of the path integral) plaquettes. For these examples, the boundary of X does not require any special treatment.

10 The symmetry group

A symmetry operator and its inverse, together with the identity operator,[23] form a group with respect to composition of operators. This is typically part of a larger group of symmetry operators $U_h(X)$ that are indexed by an element h of an abstract group H called the **symmetry group** and that satisfy the **fusion rule**[24,25]

$$\tau\big(U_h(X), U_{h'}(X)\big) \overset{\mathcal{M}}{=} U_{hh'}(X). \tag{3}$$

If h is the identity element of H, then $U_h(X)$ is the identity operator. This implies

$$U_{h^{-1}}(X) \overset{\mathcal{M}}{=} \big(U_h(X)\big)^{-1}.$$

In a d-dimensional spacetime with trivial topology, if X has fewer than $d-1$ dimensions, then H must be abelian.[26,27]

[23] The identity operator always qualifies as symmetry operator.

[24] Iqbal (2024), text around equations (V.1)-(V.3); Bhardwaj *et al* (2024), text around equation (2.44)

[25] When the manifold X occupies only a single time (instead of being extended in time), the left side of this equation reduces to the ordinary algebraic product of operators on a Hilbert space (Bhardwaj *et al* (2024), text after equation (2.20)).

[26] Gaiotto *et al* (2015), text after equation (3.1); Bhardwaj *et al* (2024), text below equation (2.45)

[27] If spacetime has nontrivial topology, then H can be nonabelian even if $\dim(X) \leq d-1$ (Gaiotto *et al* (2015), text after equation (3.1) and appendix F; Freed *et al* (2007), section 3). One example in the cited sources involves electric and magnetic fluxes on closed surfaces ($\dim(X) = 2$) for a $U(1)$ gauge field in a spacetime with topology $\mathbb{R}\times\mathbb{R}\mathrm{P}^3$ (example after equation (3.37)). The example uses the fact that the second integer cohomology group of this space has torsion (article 28539). In less sophisticated terms, it uses the fact that $\mathbb{R}\mathrm{P}^3$ (denoted $S^3/\mathbb{Z}_2$ in appendix F) has closed surfaces that are neither contractible nor orientable. The fact that electric and magnetic fluxes on non-closed surfaces don't commute with each other when their boundaries are linked is elementary (article 44135), but that isn't an example of a nonabelian H because equation (3) only uses one surface X.

11 p-form symmetries

Let $U(M_{\text{sym}})$ be a symmetry operator localized on an oriented submanifold M_{sym} of spacetime that may have a boundary, and let M_{chg} be a submanifold that does not have a boundary. An operator $\mathcal{O}(M_{\text{chg}}) \in \mathcal{M}$ is called **charged** with respect to that symmetry if[28]

$$\tau\big(U(X'), \mathcal{O}(M_{\text{chg}})\big) \overset{\mathcal{A}}{\neq} \tau\big(U(X), \mathcal{O}(M_{\text{chg}})\big) \tag{4}$$

for some pair of submanifolds X and X' that do not intersect M_{chg} and that can both be obtained from M_{sym} by continuous deformations that preserve its boundary (if any) and its orientation.

The inequality (4) implies that the process of continuously deforming X to X' (preserving the boundary) must involve intermediate manifolds that do intersect M_{chg}, even though X and X' do not.[29] Section 14 will show that in d-dimensional spacetime, this is only possible if

$$\dim(M_{\text{sym}}) + \dim(M_{\text{chg}}) \geq d - 1. \tag{5}$$

For this reason, the symmetry defined by $U(M_{\text{sym}})$ is called a ***p*-form symmetry**, where p is the minimum value of $\dim M_{\text{chg}}$ allowed by the inequality (5).[30] The name comes from the fact that the charged operators $\mathcal{O}(M_{\text{chg}})$ are often constructed by integrating a p-form over the submanifold M_{chg}. The case $p = 0$ includes the traditional concept of a global symmetry in quantum field theory.[31] Cases with $p \geq 1$ are called **higher-form symmetries**.

[28]With this relatively relaxed definition, the sum of a charged operator and an uncharged operator would still be called *charged*. The definition can be made more specific by decomposing the action of the symmetry transformation into irreducible representations of the symmetry group (Luo *et al* (2024), equation (2.1); Schafer-Nameki (2023), equation (2.10); Harlow and Ooguri (2021), equation (8.12) (using the canonical formulation)). Then some operators have a well-defined **charge** (which may be zero, so uncharged operators are a special case) and some do not.

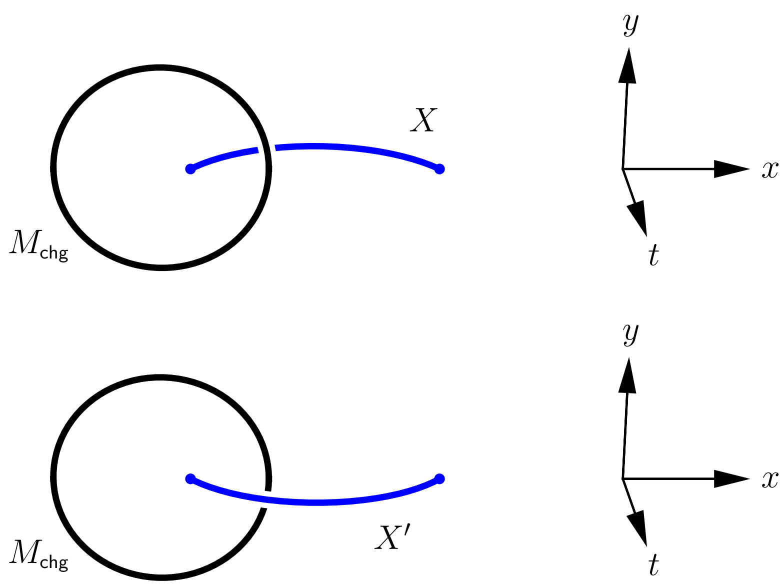
[29]This follows from the premise that $U(M_{\text{sym}})$ is a topological operator (sections 8 and 9).

[30]Gaiotto *et al* (2015); reviewed in Harlow and Ooguri (2021), section 8

[31]Sections 24-25 will illustrate this.

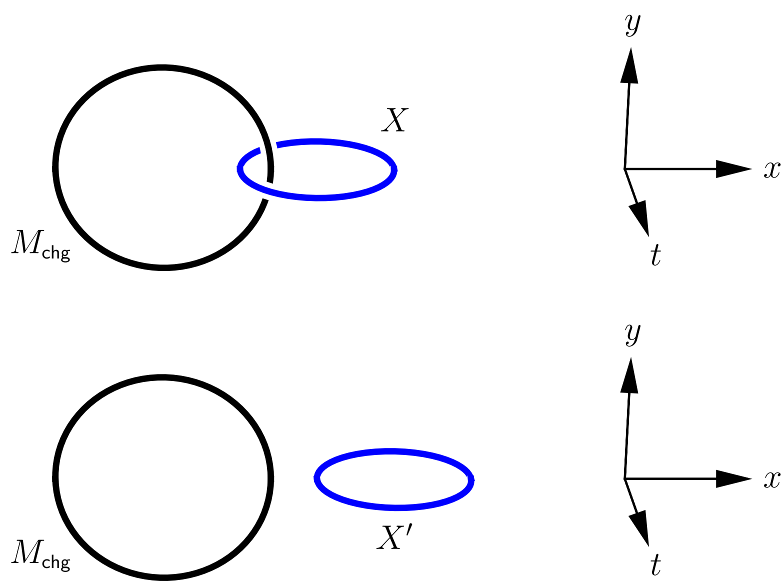
12 Picture of an M_{sym} with a boundary

These pictures illustrate a setup in which the left and right sides of (4) can be unequal. The two pictures represent the left and right sides of (4). The manifolds X and X' (which are boundary-preserving deformations of M_{sym}) are shown in blue, and M_{chg} is shown in black. The dimensions are $d = 3$, $\dim(M_{\text{sym}}) = 1$, $\dim(M_{\text{chg}}) = 1$. In this example, M_{sym} has a boundary.



13 Picture of an $M_{\textsf{sym}}$ without a boundary

These pictures illustrate another setup in which the left and right sides of (4) can be unequal. The two pictures represent the left and right sides of (4). The manifolds X and X' (which are boundary-preserving deformations of M_{sym}) are shown in blue, and M_{chg} is shown in black. The dimensions in this example are the same as in section 12, but now M_{sym} does not have a boundary.



14 The dimension condition (5)

To understand why the dimension condition (5) is a necessary condition for the inequality (4), let M_{fill} be the union of all the submanifolds in the continuous series of submanifolds that traces out the deformation from X to X'. In more precise terms, M_{fill} is the image of a continuous map h from $M_{\text{sym}} \times I$ into spacetime, where I is the interval $[0,1] \subset \mathbb{R}$, with $h(0) = X$ and $h(1) = X'$. If the dimension condition (5) were not satisfied, then $\dim(M_{\text{fill}}) + \dim(M_{\text{chg}})$ would be less than d. That implies that we could choose the homotopy h so that M_{fill} does not intersect M_{chg},[32] and then then the premise that $U(M_{\text{sym}})$ is a symmetry operator implies that the left and right sides of (4) must be equal.[33] This shows that the inequality (4) is possible only if the inequality (5) holds.

Here are a few examples of the manifold M_{fill} in the previous paragraph. In these examples, the dimension condition (5) is satisfied.

- In the example shown in section 12, we can take the manifold M_{fill} to be a disk pierced once by M_{chg}. The boundary of the disk is the circle formed by connecting the segments X and X' in that picture at their shared boundary (their shared endpoints in this case), after reversing the orientation of one so the circle has a single orientation consistent with the orientation of the disk.

- In the example shown in section 13, we can take the manifold M_{fill} to be topologically a hollow cylinder whose curved surface is pierced once by M_{chg}. The boundary of the cylinder has two components, which are the circles X and X' shown in the picture (again oriented to be consistent with the cylinder's orientation).

- Another possibility is that either X or X' could be the empty set, in which case the other one is the whole boundary of M_{fill}. An example of this is obtained by deleting the circle X' from the picture in section 13, in which case we can take M_{fill} to be a disk whose whole boundary is X.

[32] This is intuitively clear when $d \leq 3$.

[33] Footnote 29 in section 11

15 p-form symmetries at a single time

In the path integral formulation, the algebraic product of two operators that are both localized at the same time is implemented by displacing one of the operators slightly into the future or past of the other one so that their time order matches the desired algebraic order.[34] This section uses that idea to express the single-time limit of the inequality (4) using ordinary algebraic products.

Let M_s denote the spatial manifold at a single time.[35] Let M_chg be a submanifold of M_s without a boundary, and let M_sym be a submanifold of M_s that is homeomorphic to an n-dimensional ball including its boundary. Suppose those manifolds satisfy these conditions:

- $\dim(M_\text{chg}) + \dim(M_\text{sym}) = \dim(M_\text{s})$,
- M_chg and M_sym intersect each other only at isolated points (or not at all),
- M_chg does not intersect ∂M_sym.

Let $U(M_\text{sym})$ be an invertible topological operator localized on M_sym, and let X and X' be manifolds obtained by bending M_sym slightly into the future and past, respectively, without changing its boundary, so that X and X' no longer intersect M_chg. Then both sides of (4) become time-ordered products, so the condition (4) may be written[36]

$$U(X')\mathcal{O}(M_\text{chg}) \neq \mathcal{O}(M_\text{chg})U(X), \tag{6}$$

using ordinary algebraic products. In the limit where the bending is arbitrarily slight, the manifolds X and X' approach M_sym, so the condition becomes

$$U(M_\text{sym})\mathcal{O}(M_\text{chg}) \neq \mathcal{O}(M_\text{chg})U(M_\text{sym}). \tag{7}$$

This is a single-time limit of the inequality (4), expressed using ordinary algebraic products.

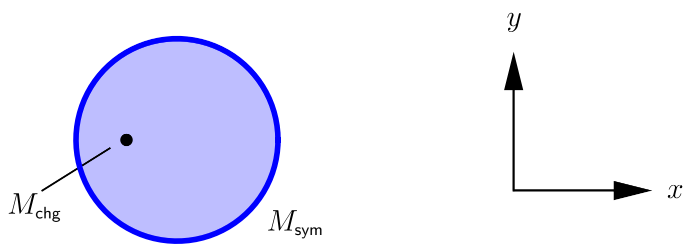
[34] Article 63548

[35] Section 2

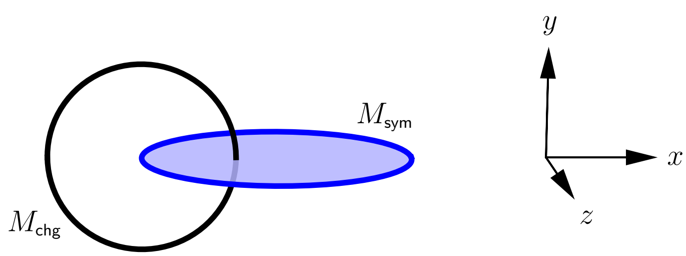
[36] The undecorated symbol $\neq$ is unambiguous in this case because the products are algebraic products, so the operators are understood to be regarded as elements of $\mathcal{A}$.

16 Pictures of single-time arrangements

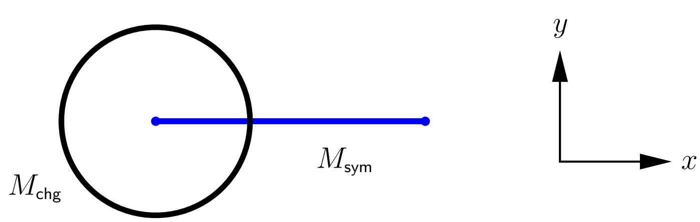
These pictures show three different configurations of manifolds M_{chg} and M_{sym} satisfying the conditions in section 15. The first example illustrates the case $\dim(M_{\text{s}}) = 2$, $\dim(M_{\text{chg}}) = 0$, $\dim(M_{\text{sym}}) = 2$:



The manifold M_{sym} is shaded blue, and its boundary is the dark blue curve. The next example illustrates the case $\dim(M_{\text{s}}) = 3$, $\dim(M_{\text{chg}}) = 1$, $\dim(M_{\text{sym}}) = 2$:



The next example illustrates the case $\dim(M_{\text{s}}) = 2$, $\dim(M_{\text{chg}}) = 1$, $\dim(M_{\text{sym}}) = 1$:



The manifold M_{sym} is the blue line, and its boundary is the pair of points marked by the blue dots.

17 The effect of a symmetry on a charged operator

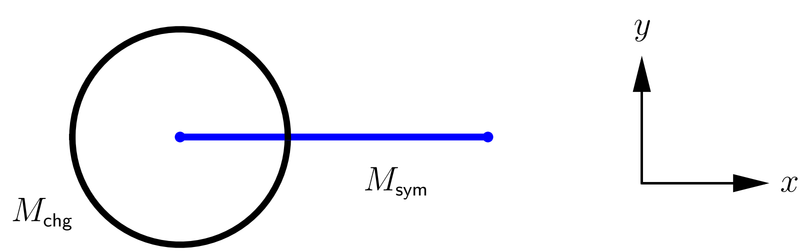
The inequality (7) may also be written

$$U(M_{\text{sym}})\mathcal{O}(M_{\text{chg}})U^{-1}(M_{\text{sym}}) \neq \mathcal{O}(M_{\text{chg}}). \tag{8}$$

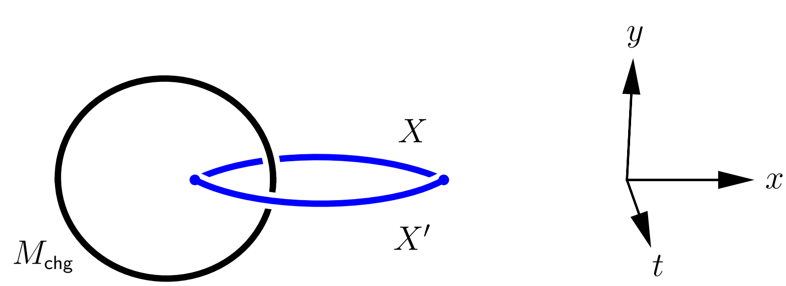
This expresses the symmetry as an automorphism of the operator algebra (whose effect on $\mathcal{O}(M_{\text{chg}})$ is nontrivial) instead of as an automorphism of the Hilbert space. Thanks to equation (2), the inequality (8) may be viewed as the single-time version of

$$\tau\big(U(X' \cup X^{-1}), \mathcal{O}(M_{\text{chg}})\big) \overset{\mathcal{A}}{\neq} \mathcal{O}(M_{\text{chg}}). \tag{9}$$

For an example, suppose spacetime is three-dimensional ($d = 3$) and that M_{sym} and M_{chg} are both one-dimensional. In the inequality (8), choose M_{sym} and M_{chg} as depicted here:



Then the left side of the corresponding inequality (9) is as depicted here:



18 Symmetry operators on manifolds with different boundaries

Section 17 explained that if $U(\cdot)$ is a symmetry operator and X and X' share the same boundary, then

$$U(X')U^{-1}(X) = U(X' \cup X^{-1}).$$

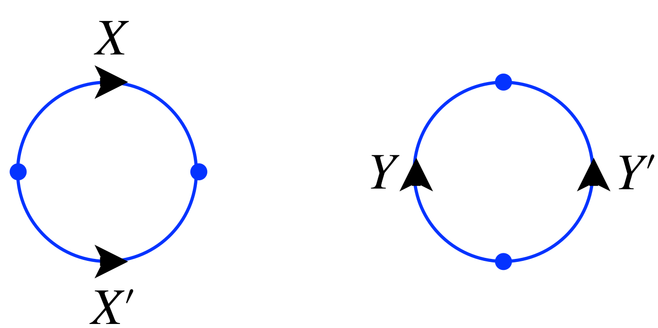
This motivates thinking of $U(X)$ and $U(X' \cup X^{-1})$ as belonging to the same family of topological operators, even though X has a boundary and $X' \cup X^{-1}$ does not.

The same closed manifold $X' \cup X^{-1}$ can also be assembled in other ways, say as

$$Y' \cup Y^{-1} = X' \cup X^{-1} \tag{10}$$

where Y and Y' have the same boundary as each other but not the same boundary of X. This motivates thinking of $U(X)$ and $U(Y)$ as belonging to the same family of topological operators even though X and Y have different boundaries.

This is illustrated below using one-dimensional manifolds:



In this example, the closed manifold (10) is a circle. The shared boundary of X and X' is one pair of points, and the shared boundary of Y and Y' is a different pair of points. Arrows indicate the orientations of X, X', Y, Y'. The orientations are such that X and X' are boundary-preserving continuous deformations of each other, and so are Y and Y'. To form a closed oriented manifold, the orientations of one of the constituents in each pair must be reversed, as in (10).

19 Example of a zero-form symmetry: outline

The rest of this article will explore examples of p-form symmetries, including one detailed example of a zero-form symmetry ($p = 0$) and a few examples of one-form symmetries ($p = 1$).

Sections 20-29 treat the zero-form case will in detail, first using the path integral formulation and then using the **canonical formulation** in which the model is defined by specifying the equation of motion and equal-time commutation relations for the field operators. This example illustrates how the concept of zero-form symmetry relates to the traditional concept of symmetry. Here's an outline:

- Section 20 uses the path integral formulation to introduce the model and one of its symmetries. Spacetime is treated as a lattice so math is unambiguous and elementary.
- Sections 21-23 shows that the symmetry is a zero-form symmetry as defined in section 11. This provides one of the easiest examples of a topological operator.
- Section 24 describes the same model in the canonical formulation.
- Section 25 uses the canonical formulation of the symmetry to illustrate the inequality (8).
- Sections 26-27 derive the topological invariance property again, this time using the canonical formulation.
- Sections 28-29 use these symmetry operators to illustrate the distinction between the two types of composition mentioned in section 5.

After that detailed exposition of the zero-form symmetry example, sections 31-34 will focus on examples of one-form symmetries.

20 Massless scalar field: path integral formulation

This section reviews the path integral formulation for a model of a single free massless scalar field. This model will be used to illustrate the concept of a zero-form symmetry.

The path integral is expressed in terms of real-valued field variables $\phi(x)$, one for each point x in spacetime. Spacetime is treated as a lattice, so x is a discrete index. We can take the lattice to be periodic in the spatial directions so the number of field variables in any finite time interval is finite. The path integral has the form[37]

$$\Psi'[\phi]_{t'} \propto \int_{<t'} [d\phi]\; e^{iS[\phi]}\Psi[\phi]_t \tag{11}$$

where Ψ and Ψ' are the initial and final states, $[\phi]_t$ denotes the set of field variables at time t, the integral is over of the field variables at times in the range $\geq t$ and $< t'$, each field variable in that range is integrated from $-\infty$ to ∞. The action is

$$S[\phi] = \sum_{(x,y)} r(x,y)\big(\phi(x) - \phi(y)\big)^2 \tag{12}$$

where the sum is over links[38] and the coefficients $r(x,y)$ are real-valued. For any real number c, the action (12) is clearly invariant under the shift

$$\phi(x) \to \phi(x) + c \qquad\qquad \text{for all } x. \tag{13}$$

Sections 21-23 will show that this symmetry of the action corresponds to a zero-form symmetry of the model.

This is a model of a **free massless scalar field**. This model has two different variants.[39] In the variant that will be used here, the operators $\phi(x)$ are themselves observables, and the symmetry (13) is spontaneously broken in the infinite-volume

[37] Article 63548

[38] A **link** is an ordered pair of neighboring points in the lattice.

[39] Article 37301

limit.[40] Even though the action is invariant under (13), the path integral (11) is still well-defined because the initial state Ψ is understood to be normalizable: $\int[d\phi]\big|\Psi[\phi]\big|^2$ is finite. Such a state cannot be invariant under (13), and that's okay: (13) still a symmetry of the collection of observables, even though it's not a symmetry of any individual state.

[40] Article 37301 calls this the **frozen** variant of the model. In a different variant of the model (called the **trimmed** variant in article 37301), an operator is not considered to be an observable unless it is invariant under $\phi \to \phi + c$.

21 Construction of the symmetry operator

For any $A \in \mathcal{M}$, the result of applying the modification A to the integrand of a path integral $\int [d\phi]\, e^{iS[\phi]}\Psi[\phi]$ will be denoted $A\int [d\phi]\, e^{iS[\phi]}\Psi[\phi]$. The initial state $\Psi[\phi]$ depends only on the field variables ϕ associated with the initial time. The action $S[\phi]$ is a function only of the **link variables** $\phi(x,y) \equiv \phi(x) - \phi(y)$. For each real number c and for each link (x,y) in the time interval covered by the path integral, define an operator $U_c(x,y) \in \mathcal{M}$ by

$$U_c(x,y)\int [d\phi]\, e^{iS[\phi]}\Psi[\phi] = \int [d\phi]\, e^{iS'[\phi]}\Psi[\phi]$$

where $S'[\phi]$ is obtained from $S[\phi]$ by replacing

$$\begin{aligned} \phi(x,y) &\to \phi(x,y) - c \\ \phi(y,x) &\to \phi(y,x) + c. \end{aligned} \tag{14}$$

Think of the lattice as a special set of points in smooth spacetime and think of the links as straight line segments connecting neighboring points. Suppose the spacetime manifold is oriented, and let X be an oriented submanifold with codimension 1 in spacetime that doesn't intersect any lattice points and whose boundary ∂X (if X has a boundary) doesn't intersect any links. We can think of choosing the orientation of X as choosing which side is the "front." Define an operator

$$U_c(X) \overset{\mathcal{M}}{\equiv} \prod_{(x,y)\nearrow X} U_c(x,y) \tag{15}$$

where the right side is an abbreviation for the composition of operators $U_c(x,y)$ over all links (x,y) that intersect X from back-to-front. Section 22 will show that this is a topological operator, and section 23 will show that it generates a one-form symmetry under which the field operator is charged.

22 Derivation of the topological property

This section shows that the operator $U_c(X)$ defined in section 21 is a topological operator. Define

$$U_c(x) \overset{\mathcal{M}}{\equiv} \prod_y U_c(x,y) \tag{16}$$

where the right side is the composition of the operators $U_c(x,y)$ whose links have their first endpoint at x. Choose any point x in the spacetime lattice that is after (not at) the initial time.[41] Applying the modification $U_c(x)$ to the integrand of the path integral is equivalent to replacing every occurrence of the field variable $\phi(x)$ with $\phi(x)+c$. This has no effect on the path integral, because the integral over $-\infty < \phi(x) < \infty$ is invariant under any shift of the integration variable $\phi(x)$. This shows that when $U_c(x)$ is regarded as an element of $\mathcal{A}$, it is just the identity operator:

$$U_c(x) \overset{\mathcal{A}}{=} \text{identity operator.} \tag{17}$$

Now let x be an endpoint of any link that pierces X, and let X' be a manifold obtained from X by a continuous deformation that passes through the point x during the deformation process but not through any other points, as illustrated in figure 1. The definitions (15) and (16) imply that the sign in $U_{\pm c}(x)$ can be chosen to enforce

$$\tau\big(U_c(X),\, U_{\pm c}(x)\big) \overset{\mathcal{M}}{=} U_c(X'). \tag{18}$$

This process may be iterated to achieve any boundary-preserving continuous deformation of the manifold X that avoids the initial time. Combine this with the identity (17) to conclude that $U_c(X)$ is a topological operator as defined in section 8.

[41] This restriction ensures that only factor in the integrand that depends on $\phi(x)$ is e^{iS}, not the initial state Ψ.

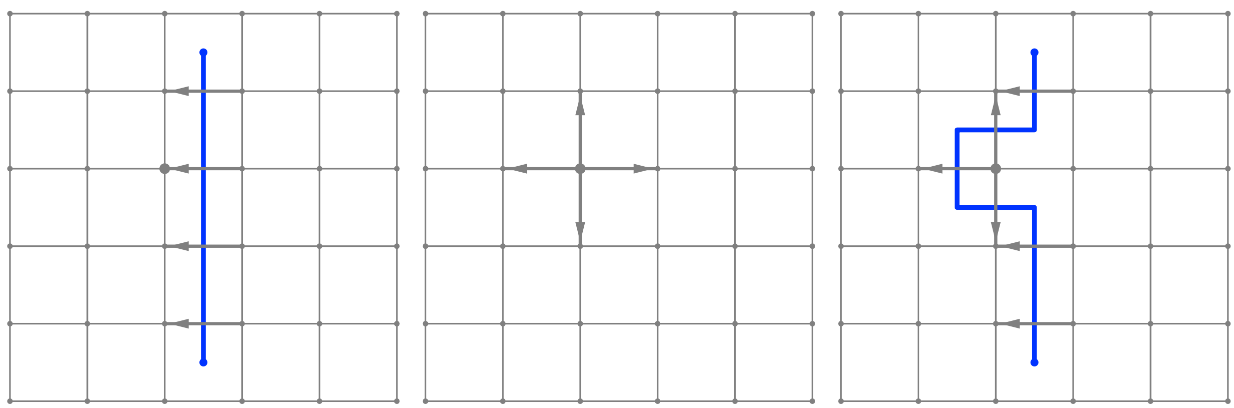


Figure 1 – These pictures illustrate the configuration in equation (18). Spacetime is 2-dimensional ($d = 2$). Gray dots are points in the lattice. Thin gray lines are links. The left picture shows the manifold X as a vertical blue line. The links involved in the composition on the right side of equation (15) are highlighted by thick gray arrows. The middle picture depicts the operator $U_{\pm c}(x)$. The links involved in the composition on the right side of equation (16) are highlighted by thick gray arrows. The large gray dot is the point x. The right picture shows the manifold X' as a blue line, now rerouted around the opposite side of the point x.

23 The zero-form symmetry

Define an operator $\phi(x) \in \mathcal{M}$ by

$$\phi(x) \int [d\phi]\, e^{iS[\phi]}\Psi[\phi] \equiv \int [d\phi]\, \phi(x) e^{iS[\phi]}\Psi[\phi].$$

In this equation, the same symbol $\phi(x)$ is used for two different things: it denotes an operator (left side) and an integration variable (right side). The operator $\phi(x)$ is called the **field operator**, and the integration variable $\phi(x)$ is called the **field variable**. This section shows that the operators $U_c(X)$ defined in section 21 generate a zero-form symmetry and that the field operator is charged with respect to that symmetry. This corresponds to the symmetry (13) of the action.

The operator $U_c(X)$ is clearly invertible: its inverse is $U_{-c}(X)$. Section 22 showed that it is also a topological operator, so it is a symmetry operator as defined in section 9.

Let X' be a continuous deformation of X, let M_{fill} be a d-dimensional volume whose boundary is $X' \cup X^{-1}$, and let x be a point in M_{fill} that is not an endpoint of any link intersected by the boundary. Then the compositions $\tau\big(U_c(X),\, \phi(x)\big)$ and $\tau\big(U_c(X'),\, \phi(x)\big)$ are both defined. We can't quite use equation (18) to morph X to X' when the factor of $\phi(x)$ is in the integrand because the composition is only defined for hypersurfaces that don't come too close to x,[42] but we can shift the integration variables without changing the value of the integral, just like we did to derive equation (17). If the factor $\phi(x)$ were absent, then we could apply shifts to all the points in M_{fill} to convert $U_c(X)$ to $U_c(X')$. If the factor $\phi(x)$ is present, then one of those shifts replaces that factor with $\phi(x) \pm c$. This gives

$$\tau\big(U_c(X),\, \phi(x)\big) \overset{\mathcal{A}}{=} \tau\big(U_c(X'),\, \phi(x) \pm c\big),$$

which is an example of the inequality (4). It says that the field operator ϕ is charged with respect to the zero-form symmetry generated by $U_c(X)$.

[42] This is a lattice version of the condition "don't intersect x" in smooth spacetime.

24 Massless scalar field: canonical formulation

The **canonical formulation** can be derived from the path integral formulation by taking a continuous-time limit.[43] This section reviews the canonical formulation without derivation. Space is treated space as a lattice to keep the math unambiguous, but smooth-space notation is used here to make the equations more recognizable.

For each point x in d-dimensional spacetime, $\phi(x)$ is an operator on the Hilbert space in the Heisenberg picture. These operators satisfy the equation of motion

$$\sum_{0\le k\le d-1} \partial_k^2 \phi(x) = 0 \tag{19}$$

and the equal-time commutation relation

$$[\phi(\mathbf{y},t), \dot\phi(\mathbf{x},t)] = i\delta^{d-1}(\mathbf{x}-\mathbf{y}), \tag{20}$$

where $\dot\phi$ is the derivative of ϕ with respect to time and $[A,B] \equiv AB - BA$. In the canonical formulation, the model is defined by declaring that the operators $\phi(x)$ satisfy (19)-(20).[44]

If c is any real number times the identity operator, then the transformation

$$\phi(x) \to \phi(x) + c \tag{21}$$

is a symmetry (in the traditional sense) of the model specified by (19)-(20): making the replacement (21) in those equations does not change them, and preserves it the original association between observables and regions of spacetime. Section 25 will explain how to express this as a *zero-form symmetry* as described in section 15.

[43] Article 63548

[44] Article 52890 uses the canonical formulation to show that these two conditions are compatible with each other, without the help of the path integral formulation.

25 The zero-form symmetry again

For simplicity, suppose that the spatial manifold M_s is closed so integrals over M_s can be defined without specifying boundary conditions. To describe a unitary operator that implements the symmetry (21), first define

$$U(V,t) \equiv \exp\left(ic\int_{\mathbf{x}\in V}\dot{\phi}(\mathbf{x},t)\right) \tag{22}$$

where V is a region of space. That operator implements a version of the symmetry (21) whose effect on operators at time t is localized within V:[45,46]

$$U(V,t)\phi(\mathbf{x},t)U^{-1}(V,t) = \begin{cases}\phi(\mathbf{x},t)+c & \text{if } \mathbf{x}\in V,\\ \phi(\mathbf{x},t) & \text{otherwise.}\end{cases} \tag{23}$$

If V is the whole spatial manifold M_s, then (22) is independent of time,[47] and it generates the symmetry (21) for all x:

$$U(M_\text{s},\cdot)\phi(x)U^{-1}(M_\text{s},\cdot) = \phi(x)+c. \tag{24}$$

The time-independence of (22) when $V = M_\text{s}$ is a special case of a more general result that will be derived in sections 26-27. That result will show that $U(V,\cdot)$ is a topological operator, so comparing equation (24) to (8) reveals that $U(V,\cdot)$ is the symmetry operator for a zero-form symmetry under which the field ϕ is charged. This shows that (21) is an example of a zero-form symmetry.

[45] To deduce this quickly, use the fact that ϕ acts like i times the derivative with respect to $\dot{\phi}$ (equation (20)).

[46] The symmetry (21) is called **splittable** because it has a local version (23) that has a nontrivial effect on some observables (Harlow and Ooguri (2021), definition 2.3). Recall that $\phi(x)$ is an observable in this model (section 20).

[47] To prove this, define $Q(t)$ to be the integral of $\dot{\phi}(\mathbf{x},t)$ over all of M_s. Use the equation of motion (19) to write $dQ(t)/dt$ in terms of the laplacian of ϕ with respect to the spatial coordinates, and then use the assumption that M_s is closed to conclude that the integral representing $dQ(t)/dt$ is zero. This shows that $Q(t)$ is independent of time. This is a special case of a result that will be derived in sections 26-27.

26 Topological invariance: canonical formulation

In equation (22), the integration domain V is a $(d-1)$-dimensional volume at a given time. The definition (22) has a generalization in which V can be any spacelike oriented submanifold of spacetime with codimension 1:[48]

$$U(V) \equiv \exp\left(ic\int_{x\in V} n\cdot\partial\phi(x)\right) \qquad n\cdot\partial \equiv \sum_{0\leq k\leq d-1} n^k\partial_k, \tag{25}$$

where n is a timelike unit vector orthogonal to V at x. Consider two such Vs, say V_1 and V_2, that share the same oriented boundary[49] and that together form the boundary ∂M_{fill} of a d-dimensional closed subset[50] M_{fill} of d-dimensional spacetime. This is illustrated in figure 2 on the next page. More precisely, if V_2^{-1} is the orientation-reversed version of V_2, then

$$\partial M_{\text{fill}} = V_1 \cup V_2^{-1}. \tag{26}$$

The hypersurfaces V_1 and V_2 are assumed to be spacelike, so the manifold M_{fill} has a **corner** at their shared boundary.[51,52] Section 27 will derive the topological invariance property

$$U(V_1) = U(V_2). \tag{27}$$

The restriction to spacelike V ensures that the field operators in the exponent all commute with each other. Section 22 used the path integral formulation to show that this restriction isn't really necessary, but it simplifies things in the canonical formulation.

---

[48]To make this unambiguous, the sign of the vector n needs to be specified. This will be done implicitly in the equivalent definition (28).

[49]Suppose that V_1 and V_2 are submanifolds of spacetime so that no parts of their boundaries are missing (section 2).

[50]Recall the convention in section 2 about the word *closed*.

[51]Article 44113 reviews the concept of a *corner* in the context of smooth manifolds.

[52]M_{fill} could also be a manifold without corner if V_1 and V_2 are both Cauchy hypersurfaces whose boundaries are empty. (This is consistent with $\partial V_1 = \partial V_2$.) That arrangement is possible because the assumed topology of spacetime (section 2) implies that each Cauchy hypersurface is a compact manifold that wraps around the spatial torus.

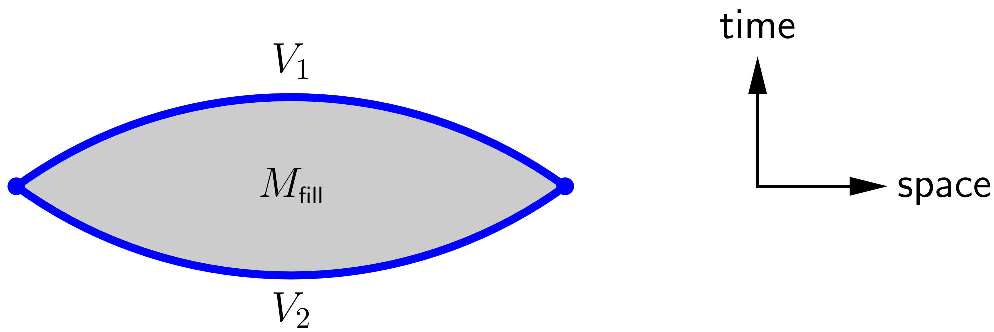


Figure 2 – Example of equation (26) in two-dimensional spacetime ($d = 2$). The region M_{fill} (shaded gray) is bounded by two spacelike hypersurfaces V_1 and V_2 (blue). The shared boundary of V_1 and V_2 is a pair of points (blue). In three-dimensional spacetime, the analogous example would be a lens-shaped region M_{fill} bounded by two disks V_1 and V_2 that bulge into the future and past, respectively, and whose shared boundary is a circle (the sharp edge of the lens). In the path integral formulation (and also in the canonical formulation after replacing the exponential in the definition (25) with a time-ordered exponential), the sharp edges can be replaced by rounded edges because V_1 and V_2 no longer need to be spacelike everywhere.

27 Topological invariance: derivation

This section uses **Stokes's theorem**[53] to show that the unitary operator $U(V)$ defined in (25) is invariant under continuous deformations[54] of V that preserve its boundary.

The definition (25) may also be written[53]

$$U(V) \equiv \exp\left(ic\int_V \star\, d\phi\right) \tag{28}$$

where $\star\, d\phi$ is the **Hodge dual** of the (operator-valued) one-form $d\phi$ with respect to the Minkowski metric on d-dimensional spacetime. In this definition, the orientation of the manifold V is understood to be specified.[53] The definition (28) is equivalent to (25) with a particular choice of the sign of the normal vector n.

The equation of motion (19) may be written $d(\star\, d\phi) = 0$, and Stokes's theorem still holds for manifolds with corners,[55] so

$$\begin{aligned}
0 &= \int_{M_{\text{fill}}} d(\star\, d\phi) && \text{(equation of motion)} \\
&= \int_{\partial M_{\text{fill}}} \star\, d\phi && \text{(Stokes's theorem)} \\
&= \int_{V_1} \star\, d\phi + \int_{V_2^{-1}} \star\, d\phi && \text{(equation (26))} \\
&= \int_{V_1} \star\, d\phi - \int_{V_2} \star\, d\phi && \text{(reverse orientation).}
\end{aligned}$$

Use this with equation (28) to get the desired result $U(V_1) = U(V_2)$.

[53] Article 91116

[54] Section 1

[55] Lee (2013), theorem 16.25

28 Composing operators: example, part 1

This section illustrates the distinction between the two types of composition in sections 4-5.

Define V_1 and V_2 as in section 26, and let M_{fill} be the region bounded by V_1 and V_2. Choose a point $x \in M_{\text{fill}}$ in the future of V_2 but in the past of V_1. The result derived in section 27 implies

$$\phi(x)U(V_1)|\psi\rangle = \phi(x)U(V_2)|\phi\rangle$$

for all states $|\psi\rangle$ and all points x in spacetime. The left side of this equation is not time-ordered.

The path integral formulation[56] automatically enforces time-ordering.[57] The time-ordered quantities are $U(V_1)\phi(x)|\psi\rangle$ and $\phi(x)U(V_2)|\phi\rangle$, and they are not always equal to each other.[58] Explicitly,[59]

$$\tau\big(U(V_1),\, \phi(x),\, U^{-1}(V_2)\big) = \begin{cases} \phi(x) + c & \text{if } x \in M_{\text{fill}}, \\ \phi(x) & \text{otherwise.} \end{cases} \tag{29}$$

Equation (29) says that $\phi(x)$ has a nonzero charge with respect to the zero-form symmetry implemented by the unitary operators (28). Equation (23) is a special case of equation (29), namely the case where the time-differences between V_1, x, and V_2 are infinitesimal.

[56] Section 20

[57] Article 02242

[58] This illustrates the inequality (6).

[59] This illustrates the inequality (9).

29 Composing operators: example, part 2

Define $\mathcal{M}$ and $\mathcal{A}$ as in section 4. The composition on the right side of equation (29) treats the operators as elements of $\mathcal{M}$, so it does not necessarily reduce to the algebraic product $U(V_1)\phi(x)U^{-1}(V_2)$ when regarded as an element of $\mathcal{A}$.[60] To understand why this is important, suppose the point x is either in the causal past of V_2 or in the causal future of V_1 (outside M_{fill}, but not spacelike to it). In that case, even though $\phi(x)$ does not commute with $U(V_1)$ and $U(V_2)$, the left side of (29) is either $\phi(x)U(V_1)U^{-1}(V_2)$ or $U(V_1)U^{-1}(V_2)\phi(x)$, and these are both equal to $\phi(x)$ because $U(V_1)$ and $U(V_2)$ are equal to each other as ordinary operators on the Hilbert space (equation (27)).

Equation (24) is true for all x. This works because (24) is expressed using the algebraic product. If x is restricted to some finite time interval, then equation (24) could also be written

$$\tau\left(U(M_{\text{s}}^{+},\cdot),\, \phi(x),\, U^{-1}(M_{\text{s}}^{-},\cdot)\right) = \phi(x) + c$$

where M_{s}^{+} and M_{s}^{-} are in that the future and past, respectively, of that time interval. To express (24) using the composition τ without any restriction on x, we can use equation (29) supplemented by the assertion that $U(V)$ is a topological operator. This is not as concise as equation (24), but it conveys more insight because equation (29) is more flexible: it does not require V_1 to be equal to V_2 or to be a Cauchy surface. It doesn't even require V_1 and V_2 to be purely spacelike.

[60] In this example, the composition τ can legitimately be called the *time-ordered product*, because $U(V)$ can be factorized into a product of point-localized operators, and these factors can be individually rearranged.

30 Relating the two formulations

This section explains how to relate the canonical representation of the symmetry operator (equation (22)) to its path integral representation (equation (15)).

In both formulations, a state is represented by a function $\Psi[\phi]$ of the field variables $\phi(x)$ at a single time, say t. In the canonical formulation, if x is a point in spacetime at time t, then the operator $\dot{\phi}(x)$ is represented as a derivative:[61]

$$\dot{\phi}(x)\Psi[\phi] = \frac{-i}{\epsilon^{d-1}}\frac{\partial}{\partial\phi(x)}\Psi[\phi]$$

where ϵ is the lattice spacing. Equivalently,

$$\exp\left(ic\epsilon^{d-1}\dot{\phi}(x)\right)\Psi[\phi] = \Psi[\phi'] \qquad\qquad \phi'(y) = \begin{cases}\phi(y)+c & \text{if } y = x,\\ \phi(y) & \text{otherwise.}\end{cases}$$

Using this representation of $\dot{\phi}$ in equation (22) reproduces equation (23) at this one value of t.[62]

Now suppose that the initial state in the path integral (11) is

$$\Psi[\phi] = U_c(V)\Psi_0[\phi] \qquad\qquad U_c(V) \equiv \exp\left(ic\int_{x\in V}\dot{\phi}(x)\right)$$

Integrate-by-parts to transfer the differential operator $U_c(V)$ from the factor $\Psi_0[\phi]$ to the factor $e^{iS[\phi]}$:

$$\int[d\phi]\,e^{iS[\phi]}U_c(V)\Psi_0[\phi] = \int[d\phi]\,\left(U_{-c}(V)e^{iS[\phi]}\right)\Psi_0[\phi] = \int[d\phi]\,e^{iS[\phi']}\Psi_0[\phi]$$

where $\phi'(x) = \phi(x) - c$ if $x \in V$ and $\phi'(x) = \phi(x)$ otherwise. The action $S[\phi]$ depends only on the differences $\phi(x,y) \equiv \phi(x) - \phi(y)$ for links (x,y), so $S[\phi']$ may

[61] Article 52890

[62] The sum $\epsilon^{d-1}\sum_{x\in V} f(x)$ is a discretized version of the integral $\int_{x\in V} d^{d-1}x\ f(x)$.

also be obtained from $S[\phi]$ by making the replacement (14) for all links that have x as an endpoint. To relate this to the path integral representation of the symmetry operator (equation (15)), define a $(d-1)$-dimensional hypersurface X like this:

- Start with the hypersurface V, which sits exactly at the initial time t.
- If the boundary of V intersects any points in the lattice, then push it slightly outward (still at time t) so that it doesn't.
- Shift the hypersurface forward in time to $t+\epsilon/2$, but drag its boundary backward in time to $t-\epsilon/2$. Call the resulting hypersurface X.

The hypersurface X intersects just the links whose link variables are shifted to obtain $S[\phi']$ from $S[\phi]$ as described above. The timelike links intersected by X have one endpoint in V at time t and the other endpoint at time $t+\epsilon$. The spacelike links intersected by X have both endpoints at time t, one inside V and one outside V. Using this X in equation (15) reproduces the representation of the symmetry operator derived above from the canonical formulation. The reasoning in section 22 can be used to show that this operator (as an element of $\mathcal{A}$) is invariant under boundary-preserving continuous deformations of X restricted to times after the initial time. The boundary of X remains at (actually slightly before) the initial time.

Beware that the path integral representation of the symmetry operator (22) is not given by replacing $\dot\phi(x)$ with the discrete version of the time derivative of the field variable $\phi(x)$. Section 35 will explain why that doesn't work.

31 Center symmetry: path integral formulation

This section reviews an example of a one-form symmetry ($p = 1$ in section 11).

The context is a model whose only field is a gauge field.[63] The gauged group G is any compact Lie group, not necessarily connected and possibly even finite, but the most commonly considered cases are $G = SU(n)$ and $G = U(1)$. Spacetime is d-dimensional with $d \geq 3$ so the gauge field has both electric (time-space) and magnetic (space-space) components.[64]

The model has a family of topological operators $T_z(\Sigma)$ called **'t Hooft operators**[65] where Σ is a $(d-2)$-dimensional submanifold of d-dimensional spacetime, possibly with a boundary, and z is any element of the center of G (which means z commutes with everything in G). If z is the identity element of G, then $T_z(\Sigma)$ is the identity operator. The model also has operators $W_r(C)$ called **Wilson operators** where C is a closed curve in spacetime and r is an irreducible representation of G.

Suppose that Σ is the boundary of a $(d-1)$-dimensional ball V in spacetime ($\Sigma = \partial V$), and suppose that Σ and C don't intersect each other. Then, in the path integral formulation,[66]

$$\tau\big(T_z(\Sigma),\, W_r(C)\big) = r(z)^{\eta(C,V)} W_r(C) \tag{30}$$

where $\eta(C, V)$ is the intersection number between C and V.[67] According to section 11, equation (30) says that if $r(z) \neq 1$, then the Wilson operators $W_r(C)$ are charged under the one-form symmetry generated by the 't Hooft operators $T_z(\Sigma)$. This one-form symmetry is called **center symmetry** because the symmetry group (section 10) is the center of G.

[63]If the model includes matter fields, then the *center symmetry* described in this section may be partially or completely broken (Sulejmanpasic and Gattringer (2019), section 2.1). It may be *spontaneously* broken even if matter fields are absent, but then it is still a symmetry as defined in sections 11 and 15.

[64]Article 31738

[65]Article 82508 constructs these operators in the path integral formulation.

[66]Section 5 defined $\tau(\cdots)$.

[67]Article 44113

32 Center symmetry: canonical formulation

Now suppose that Σ and C are both restricted to a single time, allow Σ to have a boundary, and suppose that C intersects Σ only transversely (if at all). Then, in the canonical formulation, these operators satisfy[68,69,70]

$$T_z(\Sigma)W_r(C)\big(T_z(\Sigma)\big)^{-1} = r(z)^{\eta(C,\Sigma)}W_r(C) \tag{31}$$

where $\eta(C,\Sigma)$ is the intersection number between C and Σ. This is another way to express the fact that if $r(z)\neq 1$, then the Wilson operators $W_r(C)$ are charged under the one-form symmetry generated by the 't Hooft operators $T_z(\Sigma)$.[71]

The relationship between the arrangements for the canonical formulation (31) and the path integral formulation (30) is easy to draw when spacetime is three-dimensional ($d=3$), like in section 17:

- The first picture in that section illustrates the arrangement on the left side of equation (31) when the manifolds $M_{\text{sym}}=\Sigma$ and $M_{\text{chg}}=C$ intersect each other once.
- The second picture in that section illustrates the corresponding arrangement on the left side of equation (30) with $\Sigma = X'\cup X^{-1}$. We can think of this Σ as two copies of the Σ in (31), bowed into the future and past, respectively, like the picture in section 12.

[68] Gaiotto *et al* (2017), equation (2.5) (for the case $G=SU(N)$ and $d=4$)

[69] Article 53519 constructs $T_z(\Sigma)$ in the canonical formulation and derives equation (31) for arbitrary G and d.

[70] This is analogous to equation (23).

[71] Section 15

33 Center symmetry: more examples

This section describes a few examples of arrangements on the left side of equation (31) in four-dimensional spacetime (three-dimensional space).

- **Example in which Σ has a boundary:** The middle picture in section 16 illustrates the arrangement on the left side of equation (31) when the manifold $M_{\text{sym}} = \Sigma$ is a disk that intersects the loop $M_{\text{chg}} = C$ once.
- **Example in which Σ is closed and contractible:** If Σ is the boundary of a 3-dimensional ball V in space, then $\eta(C, \Sigma) = 0$ because the intersections come in oppositely-oriented pairs that cancel each other. In this case, $T_z(\Sigma)$ is the identity operator when it is viewed as nothing more than an operator on the Hilbert space. This is consistent with equation (31). On the other hand, this same $T_z(\Sigma)$ is nontrivial when viewed as a modification of the integrand of the path integral.[72] This is consistent with equation (30), because $\eta(C, V)$ can be nonzero even though $\eta(C, \Sigma) = \eta(C, \partial V)$ is zero.
- **Example in which Σ is closed and non-contractible:** Suppose the spatial manifold is a cube with opposite faces identified. Topologically, this is a three-dimensional torus (a cartesian product of three circles). Take C to be a closed loop parallel to one of the cube's axes, so it wraps around one dimension of the torus. Take Σ to be the closed surface given by a plane orthogonal to C, so Σ is wraps around the other two dimensions of the torus. Then C intersects Σ exactly once, even though both manifolds are closed, so $\eta(C, \Sigma) = \pm 1$. In this case, equation (31) says the operators $T_z(\Sigma)$ and $W_r(C)$ don't commute with each other when $r(z) \neq 1$. This shows that $T_z(\Sigma)$ can be nontrivial as an operator on the Hilbert space even if Σ is closed.

[72] Section 4

34 One-form symmetries in electrodynamics

Consider compact electrodynamics (gauged group $U(1)$) without matter. In this context, the operators that generate the center symmetry reviewed in sections 31-33 are related to something more familiar: electric flux. In the canonical formulation, when Σ is restricted to a single time, the 't Hooft operator $T_z(\Sigma)$ is related to the electric flux operator $E(\Sigma)$ by[73,74]

$$T_z(\Sigma) = e^{i\alpha E(\Sigma)/q^2} \qquad\qquad \text{with } z = e^{i\alpha} \tag{32}$$

where q is a unit of electric charge[75] and α is a real number.[76] This is often called the **electric one-form symmetry**.[77,78]

In four-dimensional spacetime, electrodynamics also has a **magnetic one-form symmetry**[79] generated by the **topological Wilson operators**[73,80]

$$W(S) = e^{i\beta B(S)/\hbar}, \tag{33}$$

where β is a non-integer[81] real number and $B(S)$ is the magnetic flux operator on a surface S.

[73]Article 44135

[74]The group $U(1)$ is abelian, so z can be any element of $U(1)$.

[75]Article 26542 introduces the units convention used here.

[76]If α is an integer multiple of 2π, then (32) is the identity operator.

[77]McGreevy (2022), section 2.1; Brennan and Hong (2023), section 2.2.1; Benedetti *et al* (2025), section 3.2

[78]Section 2.1 in Sulejmanpasic and Gattringer (2019) calls it the **electric center symmetry**.

[79]When spacetime is d-dimensional, the magnetic symmetry is a $(d-3)$-form symmetry (Heidenreich *et al* (2021), section 3.4.2).

[80]Article 40191 constructs topological Wilson operators for nonabelian gauged groups G, but the operators constructed there are not invertible, so they don't qualify as *symmetry operators* as defined in section 9.

[81]If β were an integer, then (33) would depend only on the boundary ∂S of the surface S, so it would not generate a symmetry. This is also mentioned in Sulejmanpasic and Gattringer (2019), section 2.1. Article 40191 explains the relationship between topological and non-topological Wilson operators more generally, for any compact connected gauged group G.

35 Appendix

This section reviews two different ways to represent the operator $\dot{\phi}(x)$ in the integrand of a path integral and shows that they are not interchangeable when the operator is exponentiated as in (22). This substantiates the warning that was issued at the end of section 30.

For simplicity, consider a scalar field in zero-dimensional space, so space has only one point and the "field" consists of a single not-necessarily-harmonic oscillator. Then the path integral for a single time-step is[82,83]

$$\Psi_{t+dt}(s') \propto \int_{-\infty}^{\infty} ds\; e^{iS(s',s)}\Psi_t(s)$$

where $\Psi_t(s)$ is the state at time t as a function of the single field variables s, and the action is

$$S(s',s) = \frac{(s'-s)^2}{2\,dt} - dt\,v(s) \tag{34}$$

for some real-valued function $v(s)$ with a finite lower bound. Consider the identity

$$\begin{aligned}\int_{-\infty}^{\infty} ds\; e^{iS(s',s)}\frac{\partial}{\partial s}\Psi(s) &= -\int_{-\infty}^{\infty} ds\; \left(\frac{\partial}{\partial s}e^{iS(s',s)}\right)\Psi(s)\\ &= \int_{-\infty}^{\infty} ds\; e^{iS(s',s)}\left(i\,\frac{s'-s}{dt} + O(dt)\right)\Psi(s).\end{aligned}$$

This says that when dt is small enough, the linear operator $\Psi(s) \to \frac{\partial}{\partial s}\Psi(s)$ can be represented by inserting a factor of $i(s'-s)/dt$ into the integrand of the path integral. In contrast, when $n > 1$, the linear operator

$$\Psi(s) \to \left(\frac{\partial}{\partial s}\right)^n \Psi(s)$$

[82] Article 63548

[83] This is also the single-time-step path integral for a model of a single nonrelativistic spinless particle living in a one-dimensional space parameterized by the coordinate s.

is not represented by inserting $\big(i(s'-s)/dt\big)^n$, not even as an approximation when dt is small. This is demonstrated by the case $n=2$, where the same steps as before lead to the identity

$$\int_{-\infty}^{\infty} ds\; e^{iS(s',s)}\left(\frac{\partial}{\partial s}\right)^2\Psi(s)=\int_{-\infty}^{\infty} ds\; e^{iS(s',s)}\left(-\frac{(s'-s)^2}{dt^2}+O(dt)+\frac{i}{dt}\right)\Psi(s).$$

The constant[84] term i/dt is not negligible when dt is small. The corresponding relationship for the shift operator $\exp(r\,\partial/\partial s)$ with $r\in\mathbb{R}$ is

$$\begin{aligned}\int_{-\infty}^{\infty} ds\; e^{iS(s',s)}\exp\left(r\,\frac{\partial}{\partial s}\right)\Psi(s)&=\int_{-\infty}^{\infty} ds\; e^{iS(s',s)}\Psi(s+r)\\&=\int_{-\infty}^{\infty} ds\; e^{iS(s',s-r)}\Psi(s)\\&=\int_{-\infty}^{\infty} ds\; e^{iS(s',s)}e^{ir(s'-s)/dt}e^{ib}\Psi(s)\end{aligned}\tag{35}$$

with $b=\frac{r^2}{2\,dt}+dt\,\big(v(s-r)-v(s)\big)$. When dt is small enough to ignore the s-dependent term, the relationship (35) reduces to

$$\int_{-\infty}^{\infty} ds\; e^{iS(s',s)}\exp\left(r\,\frac{\partial}{\partial s}\right)\Psi(s)\propto\int_{-\infty}^{\infty} ds\; e^{iS(s',s)}e^{ir(s'-s)/dt}\Psi(s).\tag{36}$$

This shows that in the shift operator $\exp(r\,\partial/\partial s)$, replacing $\partial/\partial s$ with $i(s'-s)/dt$ is valid only near the continuous-time limit and only up to an overall proportionality factor e^{ib} that behaves badly as $dt\to 0$. A similar conclusion holds in d-dimensional spacetime for any d.

[84] Here, *constant* means independent of the field variables (s and s').

36 References

(Open-access items include links.)

Benedetti *et al*, 2025. "ABJ anomaly as a U(1) symmetry and Noether's theorem" *SciPost Phys.* **18**: 041, https://doi.org/10.21468/SciPostPhys.18.2.041

Bhardwaj *et al*, 2024. "Lectures on generalized symmetries" *Physics Reports* **1051**: 1-87, https://www.sciencedirect.com/science/article/pii/S0370157323003861

Brennan and Hong, 2023. "Introduction to Generalized Global Symmetries in QFT and Particle Physics" https://arxiv.org/abs/2306.00912

Córdova *et al*, 2024. "Anomalies of non-invertible symmetries in (3+1)d" *SciPost Phys.* **17**: 131, https://www.scipost.org/SciPostPhys.17.5.131/pdf

Davighi, 2025. "Generalised Symmetries in Particle Physics" https://arxiv.org/abs/2504.05960

Etxebarria, 2022. "Branes and Non-Invertible Symmetries" *Fortsch. Phys.* **70**: 2200154, https://onlinelibrary.wiley.com/doi/full/10.1002/prop.202200154

Freed *et al*, 2007. "Heisenberg Groups and Noncommutative Fluxes" *Annals of Physics* **322**: 236-285, https://arxiv.org/abs/hep-th/0605200

Gaiotto *et al*, 2015. "Generalized global symmetries" *JHEP* **2015(02)**: 172, https://doi.org/10.1007/JHEP02(2015)172

Gaiotto *et al*, 2017. "Theta, Time Reversal, and Temperature" *JHEP* **2017(05)**: 091, https://doi.org/10.1007/JHEP05(2017)091

Harlow and Ooguri, 2021. "Symmetries in Quantum Field Theory and Quantum Gravity" *Communications in Mathematical Physics* **383**: 1669-1804, https://arxiv.org/abs/1810.05338

Heidenreich *et al*, 2021. "Non-Invertible Global Symmetries and Completeness of the Spectrum" *JHEP* **2021(09)**: 203, https://doi.org/10.1007/JHEP09(2021)203

Iqbal, 2024. "Jena lectures on generalized global symmetries: principles and applications" https://arxiv.org/abs/2407.20815

Lee, 2013. *Introduction to Smooth Manifolds (Second Edition).* Springer

Luo *et al*, 2024. "Lecture Notes on Generalized Symmetries and Applications" *Physics Reports* **1065**: 1-43, https://arxiv.org/abs/2307.09215

McGreevy, 2022. "Generalized Symmetries in Condensed Matter" https://arxiv.org/abs/2204.03045

Schafer-Nameki, 2023. "ICTP Lectures on (Non-)Invertible Generalized Symmetries" https://arxiv.org/abs/2305.18296

Sulejmanpasic and Gattringer, 2019. "Abelian gauge theories on the lattice: θ-terms and compact gauge theory with(out) monopoles" *Nuclear Physics B* **943**: 114616, https://arxiv.org/abs/1901.02637

37 References in this series

Article **02242** (https://cphysics.org/article/02242):
"Localized Operators as States on Boundaries"

Article **10690** (https://cphysics.org/article/10690):
"Why Does Smearing in Time Work Better Than Smearing in Space?"

Article **22721** (https://cphysics.org/article/22721):
"Overview of Wilson Operators and 't Hooft Operators"

Article **26542** (https://cphysics.org/article/26542):
"The Free Quantum Electromagnetic Field in Smooth Space"

Article **28539** (https://cphysics.org/article/28539):
"Homology Groups"

Article **31738** (https://cphysics.org/article/31738):
"The Electromagnetic Field and Maxwell's Equations in Any Number of Dimensions"

Article **37301** (https://cphysics.org/article/37301):
"Free Massless Scalar Quantum Fields"

Article **40191** (https://cphysics.org/article/40191):
"Topological Wilson Operators in Path Integrals"

Article **44113** (https://cphysics.org/article/44113):
"Submanifolds and Boundaries"

Article **44135** (https://cphysics.org/article/44135):
"Wilson and 't Hooft Operators for the Quantum Electromagnetic Field at a Single Time"

Article **52890** (https://cphysics.org/article/52890):
"Defining Scalar Quantum Fields on a Spatial Lattice"

Article **53519** (https://cphysics.org/article/53519):
"Topological 't Hooft Operators at a Single Time"

Article **53600** (https://cphysics.org/article/53600):
"Tubular Neighborhoods"

Article **61813** (https://cphysics.org/article/61813):
"Homotopy, Homotopy Groups, and Covering Spaces"

Article **63548** (https://cphysics.org/article/63548):
"Scalar Quantum Fields on a Spacetime Lattice"

Article **82508** (https://cphysics.org/article/82508):
"Topological 't Hooft Operators in Path Integrals"

Article **91116** (https://cphysics.org/article/91116):
"A Quick Review of Differential Forms and Stokes's Theorem"