Asymptotic Freedom and the Continuum Limit of Yang-Mills Theory

Randy S

Abstract Article 89053 introduces a family of models called **Yang-Mills theories** whose only quantum field is a gauge field. The models are defined by treating spacetime as a lattice, with the understanding that we only care about what the model predicts at resolutions much coarser than the lattice scale. Many models with a nonabelian gauged group in 3- or 4-dimensional spacetime have a property called **asymptotic freedom**, which is believed to be related to the existence of a nontrivial strict continuum limit. This article uses Yang-Mills theory with gauged group $SU(N)$ to introduce that relationship. This article also clarifies some things about the small-coupling expansion (also called **perturbation theory**) in models with gauge fields.

Contents

1 Introduction **4**

2 Notation **5**

3 Outline **6**

4 Simplifying conditions **7**

5 The model on a spacetime lattice 8

6 Correlation length and mass gap 9

7 Continuum and infinite-volume limits 10

8 Adjusting parameters to reach a continuum limit 11

9 The physical length scale and continuum limits 12

10 Relationship to the renormalization group(s) 13

11 The concept of a renormalization scheme 14

12 Various renormalization schemes 15

13 Preview of results 16

14 The bare coupling 17

15 Asymptotic freedom and nontriviality 18

16 Gauge-fixed expectation values 19

17 n-point correlation functions 20

18 A lemma about correlation functions 21

19 Another lemma about correlation functions 22

20 n-point correlation functions with gauge-fixing 23

21 From large β to small coupling 25

22 A feature shared by various schemes **27**

23 Example of a renormalization scheme **28**

24 Another example of a renormalization scheme **29**

25 Perturbative renormalization **30**

26 Asymptotic freedom: renormalized version **31**

27 The beta function **32**

28 Scheme-independence of the first coefficients **33**

29 Asymptotic freedom: bare and renormalized **34**

30 Deducing the slope of $d\beta/d\epsilon$ when $d \neq 4$ **36**

31 Results when $d = 4$ **37**

32 The case $G = U(1)$ **38**

33 QED compared to QCD **39**

34 Different continuum limits when $G = U(1)$ **40**

35 Correlation length and the continuum limit **41**

36 References **43**

37 References in this series **47**

1 Introduction

When the gauged group G is connected and nonabelian, the models constructed in article 89053 – called (pure) **Yang-Mills theories** – are among the simplest examples in quantum field theory of models that are believed to have nontrivial continuum limits with Lorentz symmetry.[1,2] The evidence for the existence of such a limit is related to **asymptotic freedom**, the subject of this article.

The name *asymptotic freedom* may be used for either of two closely-related mathematical phenomena. To distinguish between them, this article calls them the *bare version* and the *renormalized version* of asymptotic freedom.

This article is mostly about the bare version of asymptotic freedom. The family of lattice models considered in this article has only one continuously adjustable dimensionless parameter, which may be expressed in terms of the **bare coupling** g that will be introduced in section 14. If an appropriate low resolution[3] prediction ω is chosen, then holding the value of ω fixed as the lattice spacing goes to zero relative to a fixed physical scale may force the value of g to approach zero. This is the **bare version** of asymptotic freedom. Calculations using a variety of methods have not found any obstructions to taking that limit when spacetime has four (or fewer) dimensions and $G = SU(N_c)$. This apparent absence of obstructions is consistent with the presumed existence of a nontrivial continuum limit.[4] In contrast, the analogous calculation in models without asymptotic freedom indicates an obstruction to such a limit.[5]

Section 26 will introduce the renormalized version of asymptotic freedom, and section 29 will show that the bare and renormalized versions imply each other.

[1] A continuum limit is called **trivial** if the model's predictions in that limit can be reproduced by a model whose action is a quadratic function of the field variables. Nontrivial continuum limits are more interesting, because they can have particles that interact with each other.

[2] The models in this article are initially defined on a spacetime lattice. Some progress toward a construction of Yang-Mills theory directly in continuous spacetime has also been made (Magnen *et al* (1993)).

[3] In this article, **low resolution** means coarse compared to the distance ϵ between neighboring points in the lattice.

[4] A full proof of the existence of such a limit is not currently known (Douglass (2004); Jaffe and Witten (2000)).

[5] Sections 32-33

2 Notation

- d = number of dimensions of spacetime
- G = the gauged group, which is always either $SU(N_c)$ or $U(1)$ in this article
- N_c = number of **colors** when the gauged group is $SU(N_c)$
- N_f = number of quark fields (**flavors**), which is zero in this article except in section 31
- $N_{\rm id}$ = trace of the identity matrix in the representation of G (section 5)
- ϵ = distance between neighboring points in the lattice (section 7)
- m = mass gap, so $1/(m\epsilon)$ is the correlation length in units of the lattice spacing (section 6)
- ω = a quantity whose value is held fixed as the continuum limit is being taken (section 11)
- S = the action (section 5)
- β = overall coefficient of action (section 5)[6]
- γ = the **beta function** (section 27)[7]
- g = the **bare coupling**, defined by writing $\beta \propto 1/g^2$
- g_R = a **renormalized coupling**, a special type of quantity that can be used for ω
- E = a characteristic energy associated with g_R or ω

[6] The symbol β is commonly used for this when spacetime is treated as a lattice. This convention is related to a long-standing convention in statistical mechanics, where β traditionally denotes the inverse temperature. Path integrals in lattice quantum field theory are mathematically similar to partition functions in statistical mechanics, and the coefficient denoted β plays a similar mathematical role in both cases even though its physical interpretation is different.

[7] The *beta function* is more often denoted β, but this article uses a different symbol to avoid a collision with another common convention (footnote 6).

3 Outline

- Sections 4-5 describe the family of models that will be studied in this article.
- Sections 6-10 review some generalities about the continuum limit.
- Sections 11-12 introduce the concept of a renormalization scheme and mention a few examples.
- Section 13 previews the results.
- Sections 14-15 introduce the bare coupling.
- Sections 16-20 clarify the interplay between correlation functions and gauge invariance.
- Section 21 relates the large-β approximation to the weak-field expansion, also called the small-coupling expansion. (This article uses the symbol β for the overall coefficient of the action, following a common convention in lattice quantum field theory. The *beta function* introduced in section 27 is also commonly denoted by the same symbol.)
- Sections 22-26 review more about renormalization schemes and introduce the renormalized version of asymptotic freedom.
- Sections 27-35 introduce the *beta function*, which is often used to express results about asymptotic freedom, review some results about asymptotic freedom in $SU(N_c)$ models, and review some results about $U(1)$ models for comparison.

4 Simplifying conditions

To avoid distracting complications, this article imposes two simplifying conditions on the model's definition:

- The lattice does not wrap back on itself.
- Observables are invariant under all gauge transformations, not just under gauge transformations that act trivially on the boundary.[8]

One advantage of using a lattice that wraps back on itself would be that it allows a model to have a discrete version of translation symmetry when the lattice has finite size. One disadvantage would be that setting up a small-coupling expansion is obstructed by the presence of **zero modes**, which are combinations of link variables that aren't constrained by the gauge fixing protocol and that also don't affect the value of the action.[9] One way to eliminate zero modes is to use *twisted boundary conditions*.[10] Another way is to overtly prune the zero modes from the definition of the path integral. The easiest way, the one used in this article, is to use a lattice that doesn't wrap back on itself at all.

[8] Article 00951

[9] Article 00951; Heller and Karsch (1985)

[10] Trottier *et al* (2002), Pérez *et al* (2017)

5 The model on a spacetime lattice

This section reviews part of the construction that article 89053 describes in more detail. Let G be a compact Lie group. When spacetime is treated as a lattice, the model is constructed entirely from G-valued **link variables** $u(\ell)$, one for each directed link ℓ in the lattice. Link variables associated with oppositely-directed links are each other's inverses. This article uses the **Wilson action**

$$S[u] = \frac{\beta}{2} \sum_{\square} \left(1 - \frac{W(\square)}{N_{\text{id}}} \right), \tag{1}$$

where each **traced plaquette variable** $W(\square)$ is the trace of the product of the link variables around the perimeter of the plaquette $\square$, with the trace defined using a faithful matrix representation of G.[11] The integer N_{id} is the trace of the matrix that represents the identity element of G. Article 89053 explains how these ingredients are used to define a quantum model in the path integral formulation. Schematically, the expectation value of a product of observables $O_1, O_2, \ldots$ is[12]

$$\langle O_1 O_2 \cdots \rangle = \frac{\int [du]\; e^{-S[u]} O_1[u] O_2[u] \cdots}{\int [du]\; e^{-S[u]}}, \tag{2}$$

where $O_k[u]$ is a representation of O_k in terms of link variables.

This model has only one continuously adjustable dimensionless parameter, the coefficient β in equation (1). Generalizations involving two or more more such parameters can also be interesting[13] but will not be considered in this article.

[11]The unqualified name *plaquette variable* is often used for the product of link variables without the trace (Montvay and Münster (1997), text after equation (3.63)).

[12]We could specify initial and final states by inserting the functions that represent them into the integrand (article 89053). If we don't do that, then (after Wick rotation) the expectation value naturally approaches the vacuum expectation value as the length of the lattice in the time direction approaches infinity. More precisely, it approaches an average expectation value over all vacuum states, if more than one vacuum state exists. Those technicalities won't be important in this article.

[13]Florio *et al* (2021)

6 Correlation length and mass gap

When the gauged group G is nonabelian, numerical calculations give compelling evidence that Yang-Mills theory admits a continuum and infinite-volume limit with nonzero interactions and a nonzero **mass gap** – a gap that separates the energies of all non-vacuum states from the energy of the vacuum state.[14,15] This section relates the mass gap to the **correlation length**, which characterizes how quickly correlations between simultaneous local observables fall off as a function of the distance between them when the state is the vacuum state. Asymptotic freedom can be studied without knowing whether the mass gap is nonzero.

The correlation length is determined by the mass gap. Let m denote the mass gap. If m is nonzero, then the longest-range correlations fall off like $\sim \exp(-m|x|)$ asymptotically, where $|x|$ is the distance between the observables.[16] This says that $1/m$ is the correlation length. If m is zero (infinite correlation length), then the correlations fall off like a power of $|x|$ instead. This relationship between the mass gap and the correlation length can be deduced using the euclidean path integral formulation.[17,18] Briefly: the euclidean "time evolution" operator is $\exp(-Ht)$, where H is the hamiltonian, so correlations between observables separated in time fall off exponentially with a minimum rate determined by m. The symmetry of euclidean spacetime then implies that the same must be true of correlations between observables separated in space. This also shows that if the correlation length $1/m$ were zero, then nothing (no particles or signals) would propagate over distances large enough for experiments to resolve. Models in which the correlation is finite or infinite in physical units can both be interesting, but models with zero correlation length are not.

[14]The name *mass gap* comes from the fact that this energy difference is equal to the mass of the lowest-mass particle (in models whose low energy states have a meaningful particle interpretation).

[15]The strong-coupling approximation shows that the correlation length is finite in units of ϵ when β is small enough (Montvay and Münster (1997), section 3.4.5), and numerical evidence indicates that it remains finite for all β (Montvay and Münster (1997), section 3.7).

[16]Article 00980 illustrates this in the model of a free scalar field.

[17]Montvay and Münster (1997), equations (1.197)-(1.201)

[18]Article 07246 explains this a little more directly in the case of scalar fields.

7 Continuum and infinite-volume limits

Write ϵ for the distance between neighboring lattice sites, and write K for the number of lattice sites across the width of the lattice, so that ϵK is the overall linear size of the lattice. The model is defined by treating spacetime as a finite lattice, but the only predictions that matter are the ones whose resolutions are much coarser than the lattice spacing ϵ and much finer than the overall size ϵK of space, because those scales are both artificial.

Let ρ be any unit of length such that the resolutions of realistic measurements are nonzero and finite when expressed in units of ρ. This will be called a **physical unit of length**.[19] We could keep K finite as long as it is very large, but for aesthetic reasons we would prefer to take two strict limits:

- a continuum limit, which means $\epsilon/\rho \to 0$ subject to the constraint $\epsilon K/\rho > 0$,
- an infinite-volume limit, which means $\epsilon K/\rho \to \infty$.

This article reviews concepts related to taking a continuum limit. Most of this article assumes that the infinite-volume limit has already been taken.

To get a useful continuum limit, we need the correlation length to be nonzero in physical units,[20] which implies that we need $m\epsilon \to 0$,[21] where m is the mass gap. The quantity $m\epsilon$ is a function of the model's parameters and is independent of ρ, so reaching such a limit may require manipulating the model's parameters in addition to manipulating ρ.

[19] This article uses a system of units in which the speed of light and Planck's constant $\hbar$ are both equal to 1, so if we have chosen a physical unit of length ρ, then we can use $1/\rho$ as a physical unit of energy.

[20] Section 6 explained why this is important.

[21] This implication follows from the fact that the correlation length in physical units is $1/(m\rho)$, so we need to keep $m\rho$ finite, which implies $(m\rho)(\epsilon/\rho) \to 0$ as $\epsilon/\rho \to 0$.

8 Adjusting parameters to reach a continuum limit

Suppose that we have taken an infinite-volume limit but have not yet taken a continuum limit. Article 00951 shows that if we adopt the simplifying conditions that were declared in section 4 and use the gauge-fixing protocol that will be reviewed in section 16, then the gauge-fixed form of the path integral has this property: taking β to be large suppresses all configurations in which nearby link variables deviate significantly from each other. This strongly suggests that taking $\beta \to \infty$ should make the correlation length become infinite in units of ϵ, which is equivalent to $m\epsilon \to 0$. The rest of this article assumes that this suggestion is correct.

Values of β at which $m\epsilon \to 0$ are called **critical points**. For the models with action (1), $m\epsilon$ is nonzero when β is small enough,[22] and it is believed to be nonzero and finite for all finite values of β when the gauged group G is nonabelian and $d \in \{3, 4\}$,[23] so this article will only explore what happens when $\beta \to \infty$.

[22]This is proved using the **strong coupling** (small β) **expansion**, which has a finite radius of convergence (Montvay and Münster (1997), section 3.4.5).

[23]When $d = 4$ and $G = U(1)$, good evidence exists for a **weakly first order phase transition** at $\beta = \beta_c \approx 1$ (Arnold *et al* (2001); Arnold *et al* (2002), mentioned in Majumdar *et al* (2004)). The value of $m\epsilon$ at such a phase transition would be $\ll 1$ but not zero, so it's not quite a critical point.

9 The physical length scale and continuum limits

Taking a continuum and infinite-volume limit involves two things: tuning the model's parameters to achieve $m\epsilon \to 0$ (section 8), and choosing how the physical unit ρ depends on those parameters. In a model where spacetime is already continuous, the choice of units would not have any affect on the model's physical content,[24] but the way we make the units behave during the process of taking a limit can affect the physical content of the result.

As an example, suppose that the mass gap is nonzero,[25] which means that the model has an infinite-volume limit in which $m\epsilon K \to \infty$. Taking an infinite-volume limit requires $\rho/(\epsilon K) \to 0$, but we are free to choose the rate at which this quantity goes to zero: we can make ρ behave however we want, because the model on a finite lattice is independent of ρ. By choosing that rate, we can make the quantity $m\rho = (m\epsilon K)(\rho/(\epsilon K))$ approach either zero or a nonzero value, whichever we prefer.[26] The quantity $m\rho$ is the mass gap in physical units.

Now suppose that taking $\beta \to \infty$ gives a continuum limit.[27] After taking the infinite-volume limit, we still have the freedom to choose how ρ depends on β. If the mass gap m is nonzero, then we can choose $\rho(\beta)$ to make $m\rho$ do whatever we want as $\beta \to \infty$:[28] we can make it remain finite, or we can make it go to zero.

Even without knowing whether the mass gap is nonzero, one thing is clear: starting with one of the lattice models described in section 5, if we take an infinite-volume limit and then take $\beta \to \infty$ to get a continuum limit, then the resulting model cannot have any remaining continuously adjustable dimensionless parameters.[29] The qualifier *dimensionless* is essential because the model might still have a characteristic scale, like a mass gap.

[24] Article 37431

[25] The model reviewed in section 5 is believed to have such a limit when G is nonabelian and $d \in \{3, 4\}$.

[26] If the model doesn't have a limit with $m\epsilon K \to \infty$, then we are stuck with $m\rho \to 0$.

[27] Section 8

[28] Section 34 will illustrate this in a model where the dependence of $m\epsilon$ on β is known (for large β), namely the model with $G = U(1)$ and $d = 3$.

[29] If the limits are taken in the opposite order, then the absence of continuously adjustable dimensionless parameters in the resulting model would no longer be evident (at least not to me), because K is an integer.

10 Relationship to the renormalization group(s)

If a model in infinite continuous spacetime has one or more continuously adjustable dimensionless parameters, then we can ask how (or whether) those parameters can be adjusted to emulate an overall change of scale. This defines a flow in the space of possible parameter-values. This flow is called the **renormalization group**.[30]

Models that look the same at all scales are called **fixed points** of the renormalization group. The models traditionally studied in relativistic quantum field theory all live in neighborhoods of the **trivial** fixed point, which is the fixed point obtained by omitting all higher-than-quadratic terms in the action. Among these models, the ones with asymptotic freedom are believed to be the ones that admit nontrivial strict continuum limits. This article only considers models that live near the trivial fixed point.[31]

After taking infinite-volume and continuum limits, some models don't have any continuously adjustable dimensionless parameters, but they also don't look the same at all scales. They have a **characteristic scale**, typically expressed in units of energy.[32] Example: when the gauged group G is nonabelian and $d \in \{3, 4\}$, the models described in section 5 are thought to have a nonzero mass gap,[33] which provides a characteristic scale.[34,35] Asymptotic freedom can be studied without knowing whether the mass gap is nonzero.

---

[30] Article 10142

[31] When $d \geq 5$ and $G = SU(N_c)$ with $N_c \leq 4$, some evidence suggests the existence of a nontrivial fixed point – not necessarily for the Wilson action (1) (Morris (2005), section 2), but for other some choice of lattice action whose continuum limit is (4). An example of such evidence is presented in De Cesare *et al* (2021). Florio *et al* (2021) lists some additional references.

[32] If we take the limit as this scale goes to 0 or ∞, then the model becomes scale invariant.

[33] Section 6

[34] The model's predictions also depend on other dimensionful quantities, like the momenta of the particles entering a scattering event, but those momenta are properties of the initial state, not properties of the model itself. The characteristic scale is a property of the model itself.

[35] The original lattice model has one dimensionless continuously adjustable dimensionless parameter, namely β. After taking the infinite-volume and continuum limits, the resulting model has no continuously adjustable dimensionless parameters but does have a characteristic scale. This phenomenon is called **dimensional transmutation**.

11 The concept of a renormalization scheme

Suppose that we have already taken $K \to \infty$. Taking a continuum limit means choosing a physical unit of length, denoted ρ, and sending $\epsilon/\rho \to 0$.[36] To specify the limit unambiguously, we need to specify a relationship between β and ϵ/ρ. We can do that by choosing a function $\omega(\beta, \epsilon/\rho)$ that depends on both β and ϵ/ρ and requiring that ω remain constant as $\epsilon/\rho \to 0$. This is called choosing a **renormalization scheme**.[37] It implicitly specifies β as a function of ϵ/ρ, which will be denoted $\beta(\epsilon/\rho)$ in this article.[38]

Physically, the function $\omega(\beta, \epsilon/\rho)$ should be related to a family of meaningful low resolution predictions that is parameterized by the physical scale ρ in the continuum limit.[39] Section 12 will list a few examples of renormalization schemes that are used when $d = 4$ and $G = SU(N_c)$, and section 22 will highlight a shared feature of those schemes.

[36]Section 7

[37]The word *renormalization* is used for a few related-but-different things. The way it's used in this section could be called *nonperturbative renormalization*, because its definition doesn't refer to any small-parameter expansion. In contrast, *perturbative renormalization* is a way of constructing useful small-parameter expansions (section 25).

[38]The function denoted $\beta(\epsilon/\rho)$ in this article is distinct from what is traditionally called the *beta function* in quantum field theory (section 27).

[39]The relationship between those physical predictions and ω may be indirect, and this freedom can be used to help make calculations easier.

12 Various renormalization schemes

We can specify the function $\beta(\epsilon/\rho)$ by specifying a low resolution quantity ω whose value is kept fixed as $\epsilon/\rho \to 0$. When $G = SU(N_c)$ and $d = 4$, quantities that have been used in the literature for that purpose include:

- quantities expressed in terms of n-point functions of the gauge field,[40]
- quantities expressed in terms of the expectation value of a Wilson loop,[41]
- a smoothed version of trace$(F_{ab}^2(x))$, using a recipe called the **gradient flow** to do the smoothing.[42,43]

Section 22 will highlight a feature shared by all these approaches.

Another way to specify the function $\beta(\epsilon/\rho)$ is to find a modified action on a coarser lattice that gives the same low resolution predictions as the original action on the finer lattice.[44] That approach is more difficult and is not normally used in practice, at least in models with gauge fields.[45]

[40] Sections 17-20 will explain how these functions are defined, and section 24 will describe this type of scheme in more detail.

[41] Section 23 will review this scheme.

[42] Lüscher (2010), equation (1.4) and the surrounding text, and equations (2.1), (2.14), and (3.1)

[43] Makino *et al* (2018) explains how this approach relates to the zooming-in/out idea used in article 10142. The smoothing effect is related to the one described more explicitly in Morningstar and Peardon (2004). Fodor *et al* (2012) describes a variation of the method, using finite volume to eliminate complications due to zero modes.

[44] For a family of models involving only scalar fields, article 22212 uses a similar idea that works by integrating out high-momentum modes in the path integral.

[45] Balaban (1987) explains how to implement it in Yang-Mills theory with the action (1). Theorem 2 in Balaban (1987) (for $d = 4$, and the text above it for $d = 3$) gives the form of upper and lower bounds on the (inverse of the) renormalized coupling defined using this procedure.

13 Preview of results

In a continuum limit, the correlation length must become infinite in units of the lattice spacing.[46] Section 8 established that this requirement can be satisfied by taking $\beta \to \infty$. For $G = SU(N_c)$ this appears to be the only way to satisfy that requirement, so a key question about any given renormalization scheme is whether it makes $\beta \to \infty$. For $G = SU(N_c)$, the result of applying any of the standard renormalization schemes depends on d:

- For $d \leq 4$, it drives β toward ∞ as $\epsilon/\rho \to 0$, as desired.
- For $d \geq 5$, it drives β toward zero.[47]

The $d \neq 4$ cases will be deduced in section 30. The $d = 4$ case requires additional calculations, the results of which will be reviewed in section 31.

[46] Section 7

[47] For $d \geq 5$, instead of imposing any of the standard renormalization schemes, we could independently send both $\epsilon/\rho \to 0$ and $\beta \to \infty$. That would give a continuum limit with nonzero correlation length, but the resulting model would be trivial (no interactions).

14 The bare coupling

The overall coefficient β is the only continuously adjustable parameter in the action (1). Calculations are often done by writing β in terms of the bare coupling g that was mentioned in section 1 and then absorbing a factor of g into the gauge field so that an expansion in powers of g is essentially an expansion in powers of the gauge field.[48] This section introduces the bare coupling g, using the small-g expansion of the action itself to motivate the details of the definition.

Let G be a (faithful matrix representation of a) compact Lie group. Let $T_1, T_2, ...$ be a complete set of generators for the Lie algebra, chosen to satisfy

$$\text{trace}(T_j T_k) = -\nu \delta_{jk}$$

for some $\nu > 0$.[49] Define the bare coupling $g > 0$ and the gauge field $A_a(x)$ by writing the coefficient β and the link variables $u(\ell)$ as

$$\beta = \frac{2N_{\text{id}}\epsilon^{d-4}}{\nu g^2} \qquad\qquad u(\ell) = \exp\big(g\epsilon A_a(x)\big), \tag{3}$$

where N_{id} is the trace of the identity matrix, and the endpoints of the link ℓ are x and $x + e_a$, using the subscript a to specify which dimension is parallel to the link. Each component $A_a(x)$ of the gauge field is a linear combination of the generators T_k. The definition of g in (3) is motivated by the fact that non-quadratic terms in the action are multiplied by extra powers of g compared to the quadratic term:[50]

$$S = \frac{-1}{4\nu} \int d^d x \ \sum_{a,b} \text{trace}(\partial_a A_b - \partial_b A_a)^2 + O(g)O(A^3) \tag{4}$$

where ∂ is a lattice version of the derivative, which includes a factor of ϵ in the denominator, and $\int d^d x \cdots$ is a lattice version of the integral over spacetime, which includes a factor of ϵ^d in the numerator.

[48] Section 21

[49] The conventions $\nu = 1$ and $\nu = 1/2$ are typically used when G is abelian and nonabelian, respectively.

[50] article 89053

15 Asymptotic freedom and nontriviality

Knowing that a model has asymptotic freedom does not automatically prove that it has a nontrivial continuum limit, but among models that live in a neighborhood of the trivial fixed point, those that have asymptotic freedom are thought to be the ones that have nontrivial continuum limits, partly because these are the models that don't show evidence of any obstruction to such a limit (section 33).

That might seem counterintuitive, because (the bare version of)[51] asymptotic freedom means that $g \to 0$ in the continuum limit, and the continuum limit would be trivial if the non-quadratic terms in the action (4) were absent. However, some effects of the non-quadratic terms in (4) – the terms that are multiplied by a positive power of g – may approach nonzero values even though the non-quadratic terms themselves approach zero.[52] The fact that $g \to 0$ in the continuum limit does not rule out the possibility that the continuum limit is nontrivial.[53]

[51] Section 1

[52] Analogy: even if $x \to 0$, a quantity of the form $x\,f(x)$ may remain nonzero, like when $f(x) = 1/x$.

[53] Example: Monte Carlo supports the coexistence of confinement and asymptotic freedom in the $d = 4$ $SU(2)$ model (Creutz (1980)).

16 Gauge-fixed expectation values

Expanding vacuum expectation values in powers of $1/\beta$ can be a useful tool for studying the limit $\beta \to \infty$. This section reviews an identity that can be used to define an expansion in powers of $1/\beta$.

A **maximal tree** may be roughly defined as a collection of (undirected) links that doesn't include any loops but that would include a loop if any other link were added to the collection.[54] Let T be the set of all directed links corresponding to the undirected links in a maximal tree. Article 00951 shows that if $I[u]$ is any gauge invariant function of the link variables, then

$$\frac{\int [du]\; e^{-S[u]}\; I[u]\cdots}{\int [du]\; e^{-S[u]}} = \frac{\int [du]'\; e^{-S[u']}\; I[u']\cdots}{\int [du]'\; e^{-S[u']}} \tag{5}$$

where $[du]'$ indicates integration only over the link variables that are not associated with links in T, and

$$u'(\ell) = \begin{cases} u(\ell) & \text{if } \ell \notin T \\ 1 & \text{if } \ell \in T. \end{cases}$$

This is a relatively simple example of **gauge fixing**, one in which the link variables associated with links in T are all constrained to be equal to 1.[55] The left-hand side of (5) will be called the **unconstrained** form, and the right-hand side will be called the **gauge-fixed** form.[56]

[54] Article 00951

[55] Baaquie (1977) compares two different gauge-fixing schemes in lattice Yang-Mills theory. One is a maximal-tree scheme as described here, and the other is a more symmetric scheme that makes calculations easier to manage. The **temporal gauge** and the **axial gauge** are close relatives of the type of gauge-fixing used here (article 00951).

[56] Analyses that appear to define correlation functions of non-gauge-invariant quantities are often implicitly using an identity like (5): they define the correlation function using a particular gauge-fixing scheme, typically without ever specifying the corresponding gauge-invariant quantity. Texts often emphasize that gauge fixing is a prerequisite for using perturbation theory, and the result derived in section 18 shows that it can also be a prerequisite for defining such correlation functions nonperturbatively. Example: Henty and Parrinello (1995) use the Landau gauge to define n-point correlation functions of the lattice gauge field, even though their computational method is nonperturbative.

17 n-point correlation functions

In much of the literature about Yang-Mills theory, the quantity that is held fixed as $\beta \to \infty$ is defined using a 3- or 4-point correlation function of the variables A_a that were introduced through equation (3).[57] This section takes a first step toward defining those correlation functions. Section 20 will give a complete definition.

The lattice model is constructed in terms of G-valued link variables $u(\ell)$. As explained in section 21, the Lie-algebra-valued quantities $A_a(x)$ are not uniquely determined by the link variables, but if we define[58]

$$\hat{A}_a(x) \equiv \hat{A}(\ell) \equiv \frac{u(\ell) - u^{-1}(\ell)}{2g\epsilon} \tag{6}$$

for the link ℓ with endpoints x and $x + e_a$, then equation (3) says that $\hat{A}_a(x)$ is equal to $A_a(x)$ to lowest order in the small-g expansion.

Correlation functions of the quantities $\hat{A}(\ell)$ are unambiguously defined in the lattice model, independently of the small-g approximation. Those correlation functions are not quite what we want, though: section 18 will show that if $\ell_1, \ell_2, ..., \ell_n$ is a list of links that don't share any endpoints with each other,[59] then the corresponding n-point correlation function – the vacuum expectation value of the product $\hat{A}(\ell_1)\hat{A}(\ell_2)\cdots\hat{A}(\ell_n)$ – is identically zero. This is a consequence of the fact that the action is invariant under gauge transformations but $\hat{A}(\ell)$ is not.[60]

If those correlation functions are zero, then what are textbooks really calculating when they claim to be calculating a nonzero 3- or 4-point function in Yang-Mills theory? Section 20 will answer that question.

[57] Section 24

[58] Equation (21) in Giusti *et al* (2001) takes the traceless part of (6). This article doesn't, because it's not appropriate when $G = U(1)$, and because it isn't strictly needed even when $G = SU(3)$: the non-traceless part (like the θ^3 term in $e^{i\theta} - e^{-i\theta}$ with $\theta \propto \text{diag}(2, -1, -1)$) is of higher order in g.

[59] If they did, then the continuum limit of the correlation function would involve the product of two As at the same point in spacetime. Such products require special treatment even in the simplest models (article 23277).

[60] The path integral defines a projection from arbitrary functions to gauge invariant functions. In this example, the projection gives zero. Section 19 will describe an example where the projection can give a nonzero result.

18 A lemma about correlation functions

This section derives a result that was used in section 17.

Let Γ be a set of distinct directed links, and suppose that at least one of the two endpoints of at least one of the links in Γ is not shared by any of the other links in Γ. Let $u(\ell_1)u(\ell_2)\cdots u(\ell_n)$ be a product of link variables, in any order, with $\{\ell_1, \ell_2, ..., \ell_n\} = \Gamma$. This section shows that the expectation value of that product is zero.

Write the path integral as

$$\big\langle u(\ell_1)u(\ell_2)\cdots u(\ell_n)\big\rangle \propto \int \prod_{k=1}^{n} \big(du(\ell_k)u(\ell_k)\big)\; f\big(u(\ell_1), ..., u(\ell_n)\big)$$

with

$$f\big(u(\ell_1), ..., u(\ell_n)\big) \equiv \int \left(\prod_{\ell\notin\Gamma} du(\ell)\right) e^{-S[u]}.$$

The quantity $f(u(\ell_1), ..., u(\ell_n))$ is gauge invariant and depends only on the link variables $u(\ell_1), ..., u(\ell_n)$. We assumed that one of those link variables, say ℓ_1, has an endpoint that is not shared by any of the others, so gauge invariance implies that the function f must be independent of $u(\ell_1)$. That gives

$$\big\langle u(\ell_1)u(\ell_2)\cdots u(\ell_n)\big\rangle \propto \left(\int \prod_{k=2}^{n} \big(du(\ell_k)u(\ell_k)\big)\; f\right)\left(\int du(\ell_1)\; u(\ell_1)\right).$$

The second factor is zero because the Haar integral over any compact Lie group has the property $\int dg\; g = 0$.[61]

[61] This is a special case of theorem 5.12 in Bröcker and tom Dieck (1985).

19 Another lemma about correlation functions

This section derives a result that will be used in section 20.

Let Γ be a set of directed links that forms a closed loop:

$$\Gamma = \big\{(x_1, x_2),\ (x_2, x_3),\ ...,(x_n, x_{n+1})\big\}$$

with $x_{n+1} = x_1$. The result that was derived in section 18 doesn't apply to this case, because endpoint of every link in Γ is shared by two links in Γ. This section shows that if G is an irreducible matrix representation of a compact Lie group, like the defining representation of $U(1)$ or $SU(N)$, then the expectation value of the product

$$p \equiv u(x_1, x_2)u(x_2, x_3)\cdots u(x_n, x_1)$$

is proportional to the identity matrix, which implies that the proportionality factor is $\big\langle\text{trace}(p)\big\rangle/N_{\text{id}}$, where N_{id} is the trace of the identity matrix.

That result can be derived by combining these two facts:

- The effect of a gauge transformation on p is $p \mapsto hph^{-1}$ where h is an element of G associated with the point x_1. This implies that the effect of the same gauge transformation on the expectation value $\langle p\rangle$ is $\langle p\rangle \mapsto h\langle p\rangle h^{-1}$.

- Everything else in the integrand of the path integral is invariant under gauge transformations, and so is the Haar measure, so the effect of this gauge transformation may be absorbed into a redefinition of the integration variables $u(x_1, x_2)$ and $u(x_n, x_1)$. This shows that $\langle p\rangle$ is invariant under gauge transformations.

Combining these two results gives $h\langle p\rangle h^{-1} = \langle p\rangle$ for all $h \in G$. If G is an irreducible representation of a compact Lie group, then this implies[62] that $\langle p\rangle$ is proportional to the identity matrix, as claimed.

[62] This isa consequence of **Schur's lemma** (Zelobenko (1973), section 20, page 56).

20 n-point correlation functions with gauge-fixing

Section 18 showed that a naïve definition of the correlation function of n factors of the gauge field A is identically zero if the factors are separated from each other in spacetime. When textbooks calculate a nonzero result for such correlation function, they're implicitly using an identity like (5), where the right-hand side looks like a correlation function of n factors of the gauge field even though the function $I[u]$ on the left-hand side is invariant under gauge transformations. This section explains how that works, using the gauge-fixing protocol that was reviewed in section 16.

As an example, suppose that the function $I[u]$ on the left-hand side of the identity (5) is a **Wilson loop**, the trace[63] of the product of link variables around some closed path in the lattice. On the right-hand side of (5), some of the link variables in the given Wilson loop might have been replaced by 1. In that case, the right-hand side doesn't look like the expectation value of a Wilson loop, and it doesn't look gauge invariant. If most of the factors in the given Wilson loop are associated with links in the maximal tree T, then the function $I[u']$ on the right-hand side might be a product of only a few link variables whose links don't share any endpoints with each other.

Conversely, suppose that the function $I[u']$ on the right-hand side of (5) is a product of n link variables $u(x_1, y_1)$, $u(x_2, y_2)$, ..., $u(x_n, y_n)$ whose links (x_k, y_k) don't share any endpoints with each other and don't belong to the tree T. The premise that the tree T is *maximal* means that it includes a (unique) path connecting any two given points in the lattice, so we can promote our product of n link variables to a closed loop C by inserting factors along the paths in T that connect y_1 to x_2, connect y_2 to x_3, and so on, finally connecting y_n back to x_1. That doesn't change the product, because all the factors we inserted are equal to 1. This shows that the gauge-fixed expectation value of the product of n separated link variables (the left-hand side of (5)) is secretly the full expectation value of a product of link variables around a closed loop (the right-hand side of (5)). Section 19 showed that

[63]The trace is necessary for making the Wilson loop gauge-invariant when G is nonabelian, but the result would be essentially the same even if we defined $I[u]$ without the trace, because the path integral automatically projects the result to its trace whenever G is an irreducible representation of a compact Lie group (section 19).

taking the trace doesn't change the result except for an overall factor of N_{id}, so the result is gauge invariant.

If the function $I[u']$ on the right-hand side of (5) is a product of n factors of the quantity $\hat{A}$ defined in (6), then

$$\langle \hat{A}(x_1,y_1)\hat{A}(x_2,y_2)\cdots\hat{A}(x_n,y_n)\rangle \propto \sum_{s_1,s_2,\ldots,s_n\in\{\pm 1\}} (s_1 s_2\cdots s_n)\big\langle u^{s_1}(x_1,y_1)u^{s_2}(x_2,y_2)\cdots u^{s_n}(x_n,y_n)\big\rangle, \quad (7)$$

so the corresponding function $I[u]$ on the left-hand side is a linear combination of 2^n different Wilson loops[64] (or 2^{n-1} if we don't separately count loops that circulate in opposite directions but are otherwise the same).

Altogether, this shows that the correlation function of n widely-separated factors of the gauge field can be nonzero when it's defined using a particular gauge-fixing scheme. This section considered only one gauge-fixing scheme, but a similar principle applies to all of them.

[64] The relationship $u^{-1}(\ell) = u(\tilde{\ell})$, where $\tilde{\ell}$ and ℓ are oppositely-directed versions of the same undirected link, implies that a different Wilson loop must be used for each term on the right-hand side of (7).

21 From large β to small coupling

We can write each G-valued link variable as

$$u(\ell) = e^{\theta(\ell)}, \tag{8}$$

where $\theta(\ell)$ is an element of the Lie algebra of G.[65] The study of asymptotic freedom uses a **small-coupling expansion**,[66,67] also called **perturbation theory**, which amounts to expanding the link variables (8) in powers of $\theta(\ell)$ everywhere in the path integral (2). This approximation is used frequently, but I have not found a complete explanation of when and why it works. The next paragraph gives a partial explanation, and then the rest of this section clarifies why the explanation incomplete.

Article 00951 explains that if the gauge-fixing scheme reviewed in section 16 is used, then configurations in which the link variables u differ significantly from 1 are strongly suppressed when β is large.[68] The condition $\theta(\ell) \approx 0$ implies $u(\ell) \approx 1$, so configurations with $\theta(\ell) \approx 0$ are still important when β is large. This gives a partial justification for using a small-θ approximation.

That partial justification isn't quite enough, though, because the condition $u(\ell) \approx 1$ does not imply $\theta(\ell) \approx 0$. The rest of this section uses the cases $G = U(1)$ and $G = SU(2)$ to emphasize that fact.

When $G = U(1)$, we can write $u(\ell) = e^{i\phi(\ell)}$ for a real-valued variable $\phi(\ell)$, and then the condition $u(\ell) \approx 1$ implies $\phi(\ell)$ is limited to neighborhoods of $2\pi n(\ell)$ for all integers $n(\ell)$, not just $n(\ell) = 0$. A careful formulation of the large-β approximation should account for all these neighborhoods. The **Villain model** does this by[69]

[65] This article assumes that G is defined using a faithful matrix representation, so $\theta(\ell)$ is a matrix of the same size as $u(\ell)$ and the definition of $e^{\theta(\ell)}$ is straightforward (article 18505).

[66] Calculations in models with more than one small coupling parameter usually involve expanding in powers of each of those parameters. Questions about how to treat different powers of different parameters consistently are avoided by using the **loop expansion** (Fradkin (2022), section 13.1).

[67] **Weak-field expansion** might be a better name.

[68] More generally, a large-β approximation is a saddle-point approximation: configurations that don't minimize the action are suppressed when β is large.

[69] Janke and Kleinert (1986)

replacing the action (1) with an action quadratic in the quantity $\phi(\ell) - 2\pi n(\ell)$ and replacing the integral over each link variable $u(\ell)$ in (2) with an integral over $\phi(\ell)$ and a sum over $n(\ell)$. When β is large, the path integral is dominated by values of $\phi(\ell)$ in neighborhoods of $2\pi n(\ell)$ for all integers $n(\ell)$. This is true both for the Villain model and for the original $G = U(1)$ model with action (1), so the predictions of these two models are expected to become indistinguishable from each other when β is large enough.[70] When $d = 3$, using a small-θ expansion without the sum over $n(\ell)$ can lead to qualitatively different predictions when β is large but finite.[71] This demonstrates that the small-θ approximation is not quite automatically implied by the large-β approximation.

Now consider the case $G = SU(2)$. In this case, the condition $u(\ell) = 1$ is equivalent to the condition $\theta(\ell) = 2\pi i\, n(\ell)U(\ell)DU^{-1}(\ell)$, where $n(\ell)$ is an arbitrary integer, D is a traceless diagonal matrix with determinant -1, and $U(\ell)$ is an arbitrary unitary matrix.[72] To construct a Villain-like model for $G = SU(2)$, the sum over the integer-valued variables $n(\ell)$ would need to be augmented by integrals over the unitary matrices $U(\ell)$. The resulting path integral might not be much easier to handle (either intuitively or mathematically) than the original path integral over the $SU(2)$-valued link variables. The difficulty is even greater for other compact Lie groups G with more complicated topologies.[73] This article will cope with that difficulty the same way published sources typically cope with it – namely by ignoring it, pretending that the large-β approximation implies the small-θ approximation.

[70] Janke and Kleinert (1986)

[71] The continuum limit (which involves taking $\beta \to \infty$) can be taken in such a way that the small-θ approximation gives the right answer, but when $d = 3$ the continuum limit can also be taken a different way that has qualitatively different properties. Section 34 reviews both of these limits.

[72] Topologically, this set of values of $\theta(\ell)$ is a series of concentric 2-spheres in $\mathbb{R}^3$ with radii proportional to $|n(\ell)|$. The case $G = U(1)$ can be described in a similar way, because the quantities $2\pi n$ form a series of concentric 0-spheres (a 0-sphere is a pair of points) in $\mathbb{R}^1$ with radii proportional to $|n|$.

[73] Article 92035

22 A feature shared by various schemes

Section 12 listed examples of quantities ω that can be held fixed as $\epsilon/\rho \to 0$, which determines how β behaves in that limit. The choice of which quantity to hold fixed is called a **renormalization scheme**. Section 23 will review one example of a renormalization scheme in more detail. This section highlights a feature shared by all of the most commonly-used renormalization schemes when $d = 4$.

In practice, the quantity ω to be held fixed is calculated using a small-coupling expansion,[74] whose relationship to the large-β approximation was partially explained in section 21. This leads to an equation for the ϵ/ρ-dependence of β in the small-coupling expansion.

Suppose that the gauge has been fixed[75] so that the small-coupling expansion is well-defined.[76] In the renormalization schemes that are used most often when $d = 4$, the quantity that was denoted ω in section 12 is denoted g_R^2, and its small-coupling expansion has the form[77,78]

$$g_R^2 = g^2 + c_4 g^4 + c_6 g^6 + \cdots . \tag{9}$$

The quantity g_R is called a **renormalized coupling**, and the renormalization scheme is defined by holding g_R fixed as $\epsilon/\rho \to 0$.

[74] This is why section 12 didn't list any renormalization scheme based on holding a two-particle scattering amplitude fixed. Even though Yang-Mills theory does have particles (called **glueballs**), their existence is not visible in a small-coupling expansion. Yamanaka *et al* (2021) studies glueball scattering using numerical methods.

[75] Section 16

[76] The series may be well-defined even if it doesn't converge. Small-coupling expansions are typically asymptotic (non-convergent) expansions.

[77] Creutz (1981), equation (2.23); Montvay and Münster (1997), equation (3.257)

[78] The coefficients c_k with $k \geq 4$ may grow without bound in the limit $\epsilon/\rho \to 0$. That's not necessarily a problem. Depending on their signs, it could just mean that g^2 is forced to approach zero in the limit $\epsilon/\rho \to 0$ if g_R^2 is held fixed (footnote 95 in section 25).

23 Example of a renormalization scheme

In one of the renormalization schemes that was listed in section 12, the quantity g_R in equation (9) is defined using a Wilson loop, specifically a rectangular Wilson loop that is arbitrarily long in the "time" direction[79] and with a variable width r in the space direction.[80] If $\langle W\rangle$ denotes the expectation value of the Wilson loop and t denotes the "duration" of the rectangle, then the quantity

$$V(r) \equiv -\lim_{t\to\infty} \frac{\log\langle W\rangle}{t}$$

can be interpreted as the interaction potential between two quarks in the limit of infinite quark mass, called the **static potential**.[81,82,83] Taking a Fourier transform with respect to r and multiplying by an appropriate power of the Fourier-conjugate variable E gives a quantity g_R of the form (9).[84] This g_R is invariant under gauge transformations, so gauge-fixing is not a prerequisite for defining it,[85] but gauge-fixing is still a prerequisite for using a small-g expansion to calculate g_R as previewed in section 22.[86]

[79] "Time" is in quotation marks because the calculations are normally done in the context of the euclidean path integral, which does not distinguish between time and space.

[80] Equation (2.7) in Celmaster and Kovacs (1984) shows how the expectation value of this Wilson loop depends on the "duration" and width of the rectangle to order $O(g^2)$ for $G = SU(N_c)$ and $d = 4$.

[81] Necco (2003), equation (2.10); Bali and Boyle (2002),e quation (27)

[82] Article 85870 uses a similar idea to study the force between two charge objects (in the limit of infinite mass) mediated by a massless scalar quantum field.

[83] Jezabek *et al* (1998) relates two versions of the static potential, one defined in momentum space and one in position space.

[84] Necco (2003), equations (2.14)-(2.15)

[85] This contrasts with schemes that use an n-point correlation function for ω: gauge-fixing is a prerequisite for defining those correlation functions (section 20).

[86] Section 5.1 in Smit (2002) summarizes the calculation, and Bali and Boyle (2002) and Peter (1997) show the details. The latter sources include an arbitrary number n_f of quark fields, but we can set $n_f = 0$ to get pure Yang-Mills theory.

24 Another example of a renormalization scheme

In another one of the renormalization schemes that was listed in section 12, the quantity g_R in equation (9) is expressed in terms of n-point functions of the gauge field.[87,88] This is similar to what is typically done in models of scalar fields,[89] and it is also standard in gauge theory, at least in quantum field theory texts that don't use a lattice. It can also be used in lattice gauge theory.[90]

Different renormalization schemes can lead to different behaviors of $\beta(\epsilon/\rho)$. In particular, when applied to the model with $G = U(1)$, keeping the connected part of a four-point function (which would be zero if the action were quadratic) fixed as $\epsilon/\rho \to 0$ would make $\beta \to 0$, whereas defining g_R using the Wilson-loop recipe that was described in section 23 would make β approach a finite nonzero constant.[91] In contrast, for $G = SU(N_c)$, those two different renormalization schemes both lead to the same result.

---

[87] Sections 17-20 explained how these functions are defined.

[88] Instead of using the full n-point function, this type of renormalization scheme typically uses the the **one-particle irreducible (1PI)** part, because this would be zero if interactions were absent. Beware that the use of the word *particle* in this name comes from perturbation theory. It doesn't necessarily refer to physical particles.

[89] Article 22212

[90] Hasenfratz and Hasenfratz (1980) (which corrects errors in Dashen and Gross (1981)) use this approach in lattice Yang-Mills theory, and Montvay and Münster (1997) reviews the results in equations (3.256)-(3.261).

[91] Smit (2002), equations (5.19)-(5.21)

25 Perturbative renormalization

Truncating the expansion (9) at a finite power of g is not necessarily a good approximation, because the coefficients c_k in the expansion may have large magnitudes.[92] In particular, the coefficient of the g^4 term may diverge when $\epsilon/\rho \to 0$.[93] Even though the small-g expansion may not be directly useful as an approximation, it is still indirectly useful:

- It can be used to infer something about the behavior of $\beta \sim \epsilon^{d-4}/g^2$ as $\epsilon/\rho \to 0$,[94] which then tells us something about the nature of the continuum limit.[95] This is the subject of sections 27 and 30-31.

- An expansion in powers of g can be rearranged into an expansion in powers of g_R, and truncating this expansion at a finite power of g_R can give a good approximation of some physically important quantities in the continuum limit even if truncating the original g-expansion was not a good approximation. This trick is called **perturbative renormalization**. Perturbative renormalization is the context for the concept of a *running coupling*, which will be introduced in section 26.

[92] Lepage and Mackenzie (1993), section 2.1

[93] Example: $c_4 \sim \log(\epsilon/\rho)$ when $d = 4$ and $G = SU(N_c)$. This is implied by the results that will be reviewed in section 31, which say that the quantity $\gamma_4 \equiv -\epsilon\, dc_4/d\epsilon$ in equation (11) is independent of ϵ/ρ.

[94] The desired behavior is $\beta \to \infty$ (section 8).

[95] For a contrived example, consider the finite series $g_R^2 = g^2 - g^4 \log(\epsilon/\rho) + g^6 \log^2(\epsilon/\rho)$. Holding this quantity fixed as $\epsilon/\rho \to 0$ forces $g \to 0$, which gives $\beta \to \infty$ if $d \le 4$. This remains true if the g^6 term is ignored, even though ignoring the g^6 term is not a good approximation when $|g^2 \log(\epsilon/\rho)| \gg 1$.

26 Asymptotic freedom: renormalized version

The preceding sections focused on what section 1 called the bare version of asymptotic freedom. In most of the physics literature, **asymptotic freedom** refers to what section 1 called the *renormalized version* of asymptotic freedom. This section introduces the renormalized version.

Exact calculations are rarely feasible in quantum field theory, so calculations are typically done by working out the first few terms in an expansion in powers of some small parameter. We can take this small parameter to be the quantity g_R that was introduced in section 22, because the value of that quantity is kept fixed in the continuum limit.

Given a one-parameter family of predictions $P(g_R, E)$ parameterized by the total energy E of the initial state, we can define a function $g_R(E)$ so that

$$P_{\text{tree}}(g_R(E), E) = P(g_R, E),$$

where $P_{\text{tree}}(\cdot, \cdot)$ is the lowest-order term in the small-g_R expansion of the function $P(\cdot, \cdot)$. By choosing an energy E_0 and expanding things in powers of $g_R(E_0)$, the first few terms in the expansion can be a good approximation for physical processes whose energy is close to E_0, where $g_R(E)$ is small.[96] The function $g_R(E)$ is called a **running coupling**. In Yang-Mills theory, the function $g_R(E)$ approaches zero as $E \to \infty$, so this small-coupling expansion becomes an increasingly good approximation at higher energies. This is the **renormalized version** of asymptotic freedom.[97]

In practice, we don't know how to calculate $P(g_R, E)$ or $g_R(E)$ exactly, but we can make progress by using perturbative renormalization:[98] we can calculate everything as a formal expansion in powers of the bare coupling g, and then we can invert the relationship (9) to express the bare coupling in terms of $g_R(E_0)$ so that everything ends up being expanded in powers of $g_R(E_0)$ instead.

[96] We can use $\rho = 1/E_0$ as the physical length scale in section 11.

[97] Even though it's defined in the context of a small-coupling expansion, the renormalized version of asymptotic freedom can be checked using numerical calculations (Creutz (1981)).

[98] Section 25

27 The beta function

In a renormalization scheme of the type described in section 22, a quantity of the form (9) is held fixed as $\epsilon/\rho \to 0$. This condition may be expressed as

$$D_\epsilon\, g_R^2 = 0, \tag{10}$$

using the abbreviation $D_\epsilon \equiv \epsilon\, d/d\epsilon$ for the dimensionless derivative. Equations (9) and (10) together define how g^2 depends on ϵ, which in turn defines how the quantity β in equation (3) depends on ϵ.

Equation (10) is implicitly a differential equation for the function $g^2(\epsilon)$. To make it explicit, use equation (9) to get

$$D_\epsilon g_R^2 = \big(1 + 2c_4 g^2 + 3c_6 g^4 + \cdots\big)\, D_\epsilon g^2 + (D_\epsilon c_4) g^4 + (D_\epsilon c_6) g^6 + \cdots$$

and then use this in (10) to get

$$\frac{1}{2} D_\epsilon g^2 = \gamma_4 g^4 + \gamma_6 g^6 + \cdots \tag{11}$$

with g-independent coefficients γ_k. The abbreviation

$$\gamma \equiv \frac{1}{2} D_\epsilon g^2 \tag{12}$$

will be used for the left-hand side of (11). The quantity (12) is called the **beta function** and is often denoted $\beta(g)$. This article is using the symbol β for a different purpose, namely for the overall coefficient of the action (1). Holding g_R^2 fixed as $\epsilon/\rho \to 0$ requires treating that parameter β as a function of ϵ/ρ, and this article uses the notation $\beta(\epsilon/\rho)$ for that function, not for the quantity (12). The quantity (12) will also be important, though: it will be used to access the sign of the slope of the function $\beta(\epsilon/\rho)$. This sign can be used to diagnose asymptotic freedom.

28 Scheme-independence of the first coefficients

Equation (9) implies that the functions $g(\epsilon)$ and $\tilde{g}(\epsilon)$ defined by two different renormalization schemes in that family are related to each other by

$$\tilde{g}^2(\epsilon) = g^2(\epsilon) + cg^4(\epsilon) + O(g^6). \tag{13}$$

This section shows that the coefficients of the first two terms in the small-coupling expansion of the quantity

$$\gamma \equiv \epsilon\frac{d}{d\epsilon}\, g^2 \tag{14}$$

are the same for all such renormalization schemes. To prove this, write

$$\gamma = \gamma_4 g^4 + \gamma_6 g^6 + O(g^8) \tag{15}$$

and use

$$\begin{aligned}
\epsilon\frac{d}{d\epsilon}\tilde{g}^2 &= \epsilon\frac{d}{d\epsilon}(g^2 + cg^4 + O(g^6)) && \text{(equation (13))}\\
&= \gamma \times (1 + 2cg^2 + O(g^4)) && \text{(equation (14))}\\
&= (\gamma_4 g^4 + \gamma_6 g^6 + O(g^8)) \times (1 + 2cg^2 + O(g^4)) && \text{(equation (15))}\\
&= \gamma_4 g^4 + \gamma_6 g^6 + 2c\gamma_4 g^6 + O(g^8) && \\
&= \gamma_4 \tilde{g}^4 + \gamma_6 \tilde{g}^6 + O(\tilde{g}^8). && \text{(equation (13))}
\end{aligned}$$

Comparing the last line to equation (13) shows that the first two coefficients in (11) are the same for two different renormalization schemes defined by fixing different quantities of the form (9).[99]

[99]The text below equation (4.1) in Fodor *et al* (2012) mentions a more general family of renormalization schemes in which the quantity held fixed is not limited to the form (9). Among renormalization schemes in that more general family, only the first coefficient is scheme-independent.

29 Asymptotic freedom: bare and renormalized

This section describes the relationship between the bare and renormalized versions of asymptotic freedom.[100] The bare version is about what happens to the bare coupling g when an appropriate prediction is held fixed as the lattice spacing ϵ goes to zero compared to a given physical scale. Expanding to only a finite power of the bare coupling might not always give a good approximation,[101] but it can lead to a good approximation when rearranged to be an expansion in powers of a renormalized coupling g_R that is more closely related to (or equal to) the given prediction and that is related to the bare coupling by a series of the form (9). The coefficients in that series are functions of the lattice spacing. For $d = 4$, the quantities g and g_R are dimensionless, so the coefficients depend on the lattice spacing ϵ only through the dimensionless combination ϵ/ρ where ρ is a given physical length scale.[102] In practice, the scale ρ is usually taken to be the inverse of an energy scale E that characterizes the physical process used to define g_R,[103] so the dimensionless combination is ϵE. The renormalized version of asymptotic freedom is about what happens to g_R as the characteristic energy E goes to infinity, or equivalently as the characteristic length $\rho = 1/E$ goes to zero.

At least to first order in the small-coupling expansion when $d = 4$, the bare and renormalized versions of asymptotic freedom imply each other. To deduce this, start with

$$g_R^2 = g^2 + f(\epsilon E)g^4 + O(g^6), \tag{16}$$

which is more explicit way of writing (9). For a given E, the ϵ-dependence of g is defined by holding g_R^2 fixed as $\epsilon \to 0$. The condition

$$\frac{d}{d\log\epsilon}\, g_R^2 = 0 \tag{17}$$

[100] Section 1 defined the bare version, and section 26 defined the renormalized version.

[101] Section 22

[102] Montvay and Münster (1997), equation (3.257)

[103] Section 23 will review a renormalization scheme in which the quantity held fixed is defined by starting with a rectangular Wilson loop and taking the Fourier transform with respect to its width in one dimension. Then the variable E is the Fourier conjugate of the width variable.

gives

$$\frac{d}{d\log\epsilon}\,g^2 = \frac{-f'g^4+O(g^6)}{1+2fg^2+O(g^4)} = -f'g^4+O(g^6). \tag{18}$$

For a given ϵ, the quantity g_R is a function of the physical energy scale E. The E-dependence of g_R^2 is given by equation (16):

$$\frac{d}{d\log E}\,g_R^2 = f'g^4+O(g^6) = f'g_R^4+O(g_R^6) \tag{19}$$

If $f'<0$, then equation (18) says that $g^2\to 0$ as $\epsilon\to 0$, and equation (19) says that $g_R^2\to 0$ as $E\to\infty$. Equations (18) and (19) may differ from each other at higher orders in the small-g expansion,[104] but at least to lowest order, this shows that the bare and renormalized versions are both are controlled by the same function $f(\epsilon E)$: asymptotic freedom occurs if the slope of that function is negative.

[104] Creutz (1983), text below equation (13.17); Smit (2002), equations (5.27)-(5.32)

30 Deducing the slope of $d\beta/d\epsilon$ when $d \neq 4$

What happens to β in the limit $\epsilon/\rho \to 0$ if a prediction of the form $g_R^2 = g^2 + O(g^4)$ is held fixed in units of the physical length ρ? Section 13 previewed the answer. This section uses a simple scaling argument to deduce the answer when $d \neq 4$.

The dimensionless expansion parameter is $\beta^{-1} \propto g^2\epsilon^{4-d}$, so the quantity g_R^2 in units of ρ is[105]

$$g_R^2\rho^{4-d} = g^2\rho^{4-d} + O(g^4) \propto \frac{1}{\beta}\left(\frac{\rho}{\epsilon}\right)^{4-d} + O(1/\beta^2)$$

with a proportionality factor independent of β and ϵ. The lowest-order term by itself would say that the function $\beta(\epsilon/\rho)$ defined by holding g_R^2 fixed as $\epsilon/\rho \to 0$ would be

$$\beta(\epsilon/\rho) \sim (\epsilon/\rho)^{d-4}. \tag{20}$$

The significance of this depends on the number d of spacetime dimensions:

- When $d < 4$, (20) would give $\beta \to \infty$ as $\epsilon/\rho \to 0$.[106] This is the bare version of asymptotic freedom as defined in section 1.[107]

- When $d > 4$, the opposite occurs: (20) says that β decreases as $\epsilon/\rho \to 0$, so the existence of a nontrivial continuum limit would require approaching a nontrivial fixed point, which is beyond the scope of this article.[108]

- For $d = 4$, the sign of the slope of the function $\beta(\epsilon/\rho)$ is not determined by the lowest-order term, so the g^4 term must be calculated before a conclusion can be reached. Section 31 will review the result of that calculation.

[105] This section writes the physical length scale as ρ for consistency with most of the preceding sections. We could use $\rho = 1/E$ for consistency with sections 23, 26, and 29. This would only change the notation, not the content.

[106] This is consistent with (but doesn't justify) neglecting the higher-order terms.

[107] When $G = U(1)$, calling this *asymptotic freedom* might be misleading, because the continuum limit of the model with $G = U(1)$ is strictly free (no interactions), not just asymptotically free. This is related to the absence of $O(g)$ terms in equation (4) when $G = U(1)$.

[108] Nontrivial fixed points probably don't exist for the Wilson action. Their possible existence for other lattice actions that have the same continuum limit is under investigation (footnote 31 in section 8).

31 Results when $d = 4$

For QCD with N_c colors and N_f flavors in four-dimensional spacetime, the coefficients in equation (11) are[109,110]

$$\gamma_4 = \frac{1}{(4\pi)^2}\frac{11N_c - 2N_f}{3} \qquad \gamma_6 = \frac{1}{(4\pi)^4}\frac{34N_c^3 - 13N_c^2N_f + 3N_f}{3N_c} \tag{21}$$

Set $N_f = 0$ to get the result for $SU(N_c)$ Yang-Mills theory without quarks:

$$\gamma_4 = \frac{1}{(4\pi)^2}\frac{11N_c}{3} \qquad \gamma_6 = \frac{1}{(4\pi)^4}\frac{34N_c^2}{3}. \tag{22}$$

More generally, when the gauged group G is connected and when N_f quark fields are included in a representation r of G, the first coefficient is[111]

$$\gamma_4 = \frac{11}{3}C(\text{adj}) - \frac{4}{3}N_f C(r) \tag{23}$$

where $C(r)$ is defined like ν was in section 14, but using the specified representation r.[112,113] In the first term, this quantity is defined using the adjoint representation. In the second term, it's defined using the representation r appropriate for the quark fields.[114] The important properties of $C(r)$ are:

- If r is the fundamental representation, then $C(r) > 0$.
- If G is nonabelian, then $C(\text{adj}) > 0$.
- If G is abelian, then $C(\text{adj}) = 0$.

Using these results in equation (11) gives asymptotic freedom when $G = SU(N_c)$.

[109] Lucini (2013), eq (2); Creutz (1983), eqs (12.15), (13.3), and (13.5); Smit (2002), eq (5.38) (for $N_f = 0$)

[110] Deur *et al* (2024) reviews calculations of the QCD running coupling (section 26) and comparisons to experiment.

[111] Peskin and Schroeder (1995), equation (16.134); Elliott *et al* (2018), theorem 5.1

[112] Peskin and Schroeder (1995), equation (15.78)

[113] $C(r)$ is the **quadratic Casimir invariant** for the given representation.

[114] This is typically the fundamental representation, so that each quark field has N_c components when $G = SU(N_c)$.

32 The case $G = U(1)$

When the gauged group is $G = U(1)$, a renormalized coupling g_R can still be defined as in section 23, but holding that g_R fixed as $\epsilon/\rho \to 0$ doesn't give a nontrivial continuum limit.[115] This negative result might be intuitively expected because the continuum action (4) is quadratic (independent of g) when $G = U(1)$.[116] It becomes less intuitive when matter is included, though, as in quantum electrodynamics, because then the continuum action is not quadratic. Still, the available mathematical evidence indicates that the continuum limit of quantum electrodynamics is necessarily trivial (no interactions).[117] That's not a problem for real-world applications, because choosing a value of the interaction strength g_R that is consistent with the real-world electromagnetic interaction is compatible with treating spacetime as a lattice that is much finer than the resolution of any realistic experiment.[118] It's not compatible with taking the lattice spacing all the way to zero in physical units, but we can take the lattice spacing to be effectively zero as far as any realistic experiment can tell.

On a spacetime lattice, the constructions of quantum electrodynamics (QED) and quantum chromodynamics (QED) are the same except for the choice of gauged group G: QED uses $G = U(1)$, and QCD uses $G = SU(N_c)$. This leads to an overall sign difference in the results that were reviewed in section 31. Section 33 will highlight one consequence of this sign difference. That consequence is a key part of why QCD is believed to have a nontrivial continuum limit and QED is not.

[115]What it does give depends on which g_R is used. For the g_R described in section 23, it doesn't give a continuum limit at all: it makes g approach a finite nonzero value (Smit (2002), equations (5.19)-(5.21)), so the correlation length doesn't diverge in units of the lattice spacing, which implies that the correlation length goes to zero in units of ρ when $\epsilon/\rho \to 0$. We can get a continuum limit by sending $g \to 0$, but that makes the model trivial (no interactions).

[116]Even though this might be intuitively expected, it is not obvious, because the original action (1) is not quadratic.

[117]Göckeler *et al* (1998)

[118]Quantum electrodynamics doesn't include any of the non-electromagnetic interactions that are known to exist, so it already starts to disagree with experiment at resolutions that are much, much too coarse to be noticeably affected by lattice artifacts (footnote 120 in section 33; McGreevy (2019), section 2.2).

33 QED compared to QCD

Consider the case $d = 4$ (four-dimensional spacetime). Use the abbreviation $x \equiv \log(\epsilon/\rho)$, so $x \to -\infty$ as $\epsilon/\rho \to 0$. In QED and QCD, the one-loop approximation to the beta-function is

$$\frac{d}{dx}\, g^2 = s g^4, \tag{24}$$

with $s > 0$ for QED, and $s < 0$ for QCD with a sufficiently small number of quark flavors.[119] The general solution of (24) is $g^2 = 1/(c - sx)$ for some constant c. For a continuum limit, we want $\epsilon/\rho \to 0$, so suppose we start with $\epsilon/\rho < 1$. The constant c should be such that $g^2 > 0$ for that initial value of ϵ/ρ. What happens as ϵ/ρ decreases depends on the sign of s:

- If $s < 0$, then $c - sx$ is an increasing function of ϵ/ρ. It was positive initially, so it remains positive, and $g^2 \to 0$ as $\epsilon/\rho \to 0$.

- If $s > 0$, then $c - sx$ is a decreasing function of ϵ/ρ. It was positive initially, so it crosses through zero at some finite value of x, say x_{LP}, where LP stands for *Landau pole*. Then $g^2 \to \infty$ ($\beta \to 0$) as $x \to x_{\text{LP}}$, which implies that the correlation length for the gauge field goes to zero.

Equation (24) is only approximate (it neglects terms of order g^6 and higher), but it suggests that reaching a nontrivial continuum limit is obstructed when $s > 0$ but not when $s < 0$. The obstruction when $s > 0$ is called a **Landau pole**, and the fact that no such obstruction exists (as far as we know) when $s < 0$ is consistent with the assertion that Yang-Mills theory with a nonabelian gauged group has a nontrivial strict continuum limit.[120]

[119] Section 31

[120] This can be used to quantify the statement that models like QED can still be very useful for physics despite the absence of a nontrivial strict continuum limit (section 32). Suppose that $g^2 = g_R^2/(1 - s g_R^2 \log(\epsilon/\rho))$, which satisfies equation (24) and is consistent with equation (9). With the values $g_R^2 \sim 0.01$ and $s \sim 0.01$, which are representative for QED (equation (23)), the condition $g^2 \ll 1$ allows $|\log(\epsilon/\rho)|$ to be as large as ~ 100, which is more than enough to allow ϵ to be much finer than any practically resolvable scale, even though we can't take ϵ/ρ all the way to zero.

34 Different continuum limits when $G = U(1)$

The example in this section illustrates that the properties of a continuum limit can depend on how the limit it taken. This example uses the *Villain model* that was mentioned in section 21.[121]

Consider the case $d = 3$ and $G = U(1)$. Let L be the correlation length (related to the mass gap m by $L = 1/m$), and define $\lambda \equiv L/\epsilon$. According to equation (3.1) in Athenodorou and Teper (2019), the dimensionless correlation length approaches[122]

$$\lambda^2 = (c\beta)^{-1}\exp(\tilde{c}\beta) \tag{25}$$

as $\beta \to \infty$, with β-independent constants c and $\tilde{c}$. When $d = 3$ and $N_{\text{id}} = 1$, equation (3) gives $\epsilon g^2 \propto 1/\beta$, so multiplying (25) by $(\epsilon g^2)^2$ gives

$$(g^2 L)^2 = (\epsilon g^2)^2\lambda^2 \propto \beta^{-3}\exp(\tilde{c}\beta). \tag{26}$$

Equation (25) shows that the continuum limit $\lambda \to \infty$ corresponds to $\beta \to \infty$. Equation (26) shows that $g^2 L \to \infty$ in that limit, so the limit may be taken in either of two ways:[123] we can keep L finite and let $g^2 \to \infty$, or we can keep g^2 finite (either nonzero or approaching zero) and let $L \to \infty$. In both cases, the resulting model has a single species of particle, with no interactions.[124] The mass of the particle is $\propto 1/L$.[124,125] The zero-mass case is quantum electrodynamics in three-dimensional spacetime without any matter.[126]

[121] The Villain β and the Wilson β are asymptotically equal for large β (Janke and Kleinert (1986), equation (8)), but those authors also warn that for $d = 4$, the model based on the Villain action is not a good approximation to the one based on the Wilson action near the critical value of β (where a phase transition occurs), which is finite for $d = 4$.

[122] Their equation (3.1) is actually for the mass gap (also equation (1.8a) in Göpfert and Mack (1982), equation (5) in Loan *et al* (2003), and equation (2.3) in Caselle *et al* (2015)), which they denote m_D. This is related to the correlation length by $L(\beta) = 1/m_D$ (Göpfert and Mack (1982), text above equation (1.10)).

[123] No matter how the limit is taken, the quantity ϵg^2 goes to zero because $\beta \propto 1/(\epsilon g^2)$. In other words, g^2 goes to zero when it is expressed in units of $1/\epsilon$.

[124] Athenodorou and Teper (2019), text below equation (3.3)

[125] Page 552 in Göpfert and Mack (1982) calls this a conjecture. I don't know why.

[126] The particle is a photon, which has spin zero when spacetime is three-dimensional (article 26542).

35 Correlation length and the continuum limit

This section derives a relation between lattice spacing and correlation length in units of ϵ that is consistent with the first two terms in the small-coupling expansion[127] when $d = 4$ and $G = SU(N_c)$, assuming that the mass gap is nonzero so that the correlation length is finite. The result, equation (32), is similar to the result for $d = 3$ and $G = U(1)$ that was shown in equation (25).

As in that section, let L be the correlation length, and define $\lambda \equiv L/\epsilon$. Equation (11) gives

$$\frac{-1}{2}\frac{dg^2}{d\log\lambda} = \gamma_4 g^4 + \gamma_6 g^6 + O\left(g^8\right). \tag{27}$$

To solve this, write it as

$$\frac{-1}{2}\frac{dg^2}{\gamma_4 g^4 + \gamma_6 g^6 + O\left(g^8\right)} = d\log\lambda. \tag{28}$$

Using the abbreviation $x \equiv 1/g^2$, this may also be written

$$\frac{1}{2}\frac{dx}{\gamma_4 + \gamma_6/x + O\left(1/x^2\right)} = d\log\lambda, \tag{29}$$

which implies

$$\frac{1}{2}\frac{dx}{\gamma_4}\left(1 - \frac{\gamma_6}{\gamma_4 x} + O\left(1/x^2\right)\right) = d\log\lambda. \tag{30}$$

Integrating both sides gives

$$\frac{1}{2\gamma_4}\left(x - \frac{\gamma_6}{\gamma_4}\log x + O(1/x)\right) = \log\lambda + \text{const.} \tag{31}$$

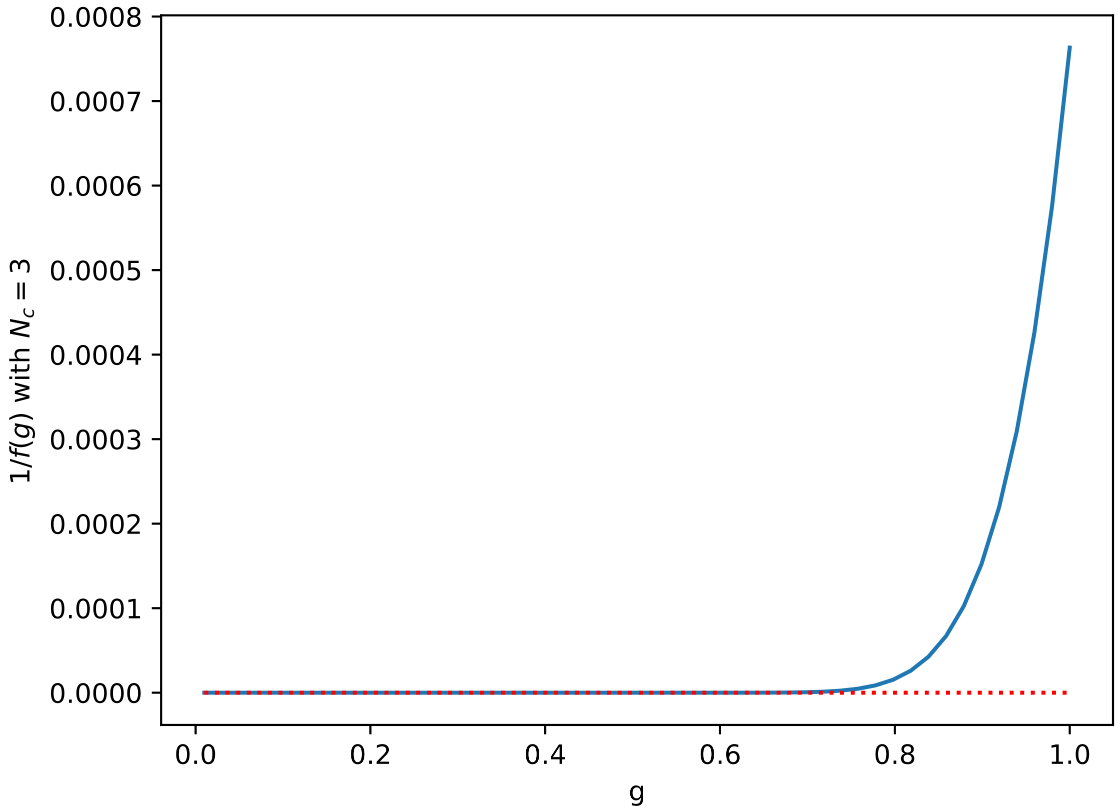
Ignoring the $O(1/x) = O(g^2)$ term, this implies[128]

$$\lambda \propto f(g) \equiv \exp\left(\frac{1}{2\gamma_4 g^2}\right) g^{\gamma_6/\gamma_4^2}. \tag{32}$$

[127] This section doesn't address whether the small-coupling expansion is a good approximation.

[128] Creutz (1983), equation (13.23); Montvay and Münster (1997), equation (3.267); Kogut (1983), equation (4.60); Allés *et al* (1998), equation (14)

Equation (32) says $g \to 0$ as $\lambda \to \infty$.[129] This is consistent with asymptotic freedom. The shape of the function $1/f(g)$ is indicated here, using the values of γ_4 and γ_6 from section 31 with $N_c = 3$ as an example:



The function $f(g)$ diverges very quickly as $g \to 0$. A few of its values are

$$f(1/2) \approx 1.6 \times 10^{12} \qquad f(1/4) \approx 2.4 \times 10^{49} \qquad f(1/8) \approx 5.6 \times 10^{198}$$

This shows how rapidly the correlation length grows in units of the lattice spacing as $g \to 0$, according to equation (32).

[129] $1/f(g) \to 0$ as $g \to \infty$, too, but only the small-g region is considered here because the derivation of (32) assumed that g is small.

36 References

(Open-access items include links.)

Allés *et al*, 1998. "Asymptotic scaling corrections in QCD with Wilson fermions from the 3-loop average plaquette" *Phys. Lett. B* **426**: 361-366, https://arxiv.org/abs/hep-lat/9801003v2

Arnold *et al*, 2001. "Finite size scaling analysis of compact QED" *Nuclear Physics B Proceedings Suplements* **94**: 651-656, https://arxiv.org/abs/hep-lat/0011058

Arnold *et al*, 2002. "Compact QED under scrutiny: it's first order" *Nuclear Physics B Proceedings Suplements* **119**: 864-866, https://arxiv.org/abs/hep-lat/0210010

Athenodorou and Teper, 2019. "On the spectrum and string tension of U(1) lattice gauge theory in $2+1$ dimensions" *JHEP* **2019(01)**: 063, https://doi.org/10.1007/JHEP01(2019)063

Baaquie, 1977. "Gauge fixing and mass renormalization in the lattice gauge theory" *Phys. Rev. D* **16**: 2612-2627, https://www.slac.stanford.edu/pubs/slacpubs/1750/slac-pub-1928.pdf

Balaban, 1987. "Renormalization Group Approach to Lattice Gauge Field Theories" *Commun. Math. Phys.* **109**: 249-301, https://projecteuclid.org/euclid.cmp/1104116842

Bali and Boyle, 2002. "Perturbative Wilson loops with massive sea quarks on the lattice" https://arxiv.org/abs/hep-lat/0210033v1

Bröcker and tom Dieck, 1985. *Representations of Compact Lie Groups.* Springer-Verlag

Caselle *et al*, 2015. "A different kind of string" *JHEP* **2015(01)**: 105, https://doi.org/10.1007/JHEP01(2015)105

Celmaster and Kovacs, 1984. "More uses for Wilson loops: Perturbation theory without Feynman diagrams" *Phys. Rev. D* **29**: 1772-1783, https://inspirehep.net/literature/193807

Creutz, 1980. "Monte Carlo study of quantized SU(2) gauge theory" *Physical Review D* **21**: 2308

Creutz, 1981. "Monte Carlo study of renormalization in lattice gauge theory" *Physical Review D* **23**: 1815-1823, https://latticeguy.net/mypubs/pub045.pdf

Creutz, 1983. *Quarks, gluons, and lattices.* Cambridge University Press

Dashen and Gross, 1981. "Relationship between lattice and continuum definitions of the gauge-theory coupling" *Physical Review D* **23**: 2340-2348

De Cesare *et al*, 2021. "Five-dimensional CFTs from the ϵ-expansion" *Physical Review D* **104**: 105015, https://arxiv.org/abs/2107.00342

Deur *et al*, 2024. "QCD Running Couplings and Effective Charges" *Progress in Particle and Nuclear Physics* **134**: 104081, https://arxiv.org/abs/2303.00723

Douglass, 2004. "Report on the Status of the Yang-Mills Millenium Prize Problem" https://www.claymath.org/library/annual_report/douglas_quantum_yang_mills.pdf

Elliott *et al*, 2018. "Asymptotic Freedom in the BV Formalism" *Journal of Geometry and Physics* **123**: 246-283, https://arxiv.org/abs/1702.05973

Florio *et al*, 2021. "Searching for continuous phase transitions in 5D SU(2) lattice gauge theory" *JHEP* **2021(12)**: 076, https://doi.org/10.1007/JHEP12(2021)076

Fodor *et al*, 2012. "The Yang-Mills gradient flow in finite volume" *JHEP* **2012(11)**: 007, https://arxiv.org/abs/1208.1051v2

Fradkin, 2022. "Chapter 13: Perturbation Theory, Regularization and Renormalization" https://eduardo.physics.illinois.edu/phys583/ch13.pdf

Giusti *et al*, 2001. "Problems in Lattice Gauge Fixing" *Int. J. Mod. Phys.* **A16**: 3487-3534, https://arxiv.org/abs/hep-lat/0104012

Göckeler *et al*, 1998. "Is there a Landau Pole Problem in QED?" *Phys. Rev. Lett.* **80**: 4119-4122, https://arxiv.org/abs/hep-th/9712244

Göpfert and Mack, 1982. "Proof of Confinement of Static Quarks in 3-Dimensional U(1) Lattice Gauge Theory for all Values of the Coupling Constant" *Commun. Math. Phys.* **82**: 545-606, https://projecteuclid.org/euclid.cmp/1103920658

Hasenfratz and Hasenfratz, 1980. "The Connection Between the Lambda Parameters of Lattice and Continuum QCD" *Physics Letters B* **93**: 165-169, https://lib-extopc.kek.jp/preprints/PDF/1980/8004/8004181.pdf

Heller and Karsch, 1985. "One Loop Perturbative Calculation of Wilson Loops on Finite Lattices" *Nucl. Phys. B* **251**: 254-278, https://inspirehep.net/literature/203758

Henty and Parrinello, 1995. "The SU(3) running coupling from lattice gluons" *Nucl. Phys. Proc. Suppl.* **42**: 264-266, https://arxiv.org/abs/hep-lat/9411085

Jaffe and Witten, 2000. "Quantum Yang–Mills Theory" https://www.arthurjaffe.com/Assets/pdf/QuantumYangMillsWebRevised.pdf

Janke and Kleinert, 1986. "How good is the Villain approximation?" *Nuclear Physics B* **270**: 135-153, https://hagenkleinert.de/documents/articles/126.pdf

Jezabek *et al*, 1998. "On the relation between QCD potentials in momentum and position space" *Phys. Lett. B* **428**: 352-358, https://arxiv.org/abs/hep-ph/9803337

Kogut, 1983. "The lattice gauge theory approach to quantum chromodynamics" *Reviews of Modern Physics, Vol.* **55**: 775-836

Lepage and Mackenzie, 1993. "On the Viability of Lattice Perturbation Theory" *Phys.Rev. D* **48**: 2250-2264, https://arxiv.org/abs/hep-lat/9209022v1

Loan *et al*, 2003. "Path Integral Monte Carlo Approach to the U(1) Lattice Gauge Theory in (2+1) Dimensions" *Phys. Rev. D* **68**: 034504, https://arxiv.org/abs/hep-lat/0209159

Lucini, 2013. "SU(N) gauge theories at large N" *Physics Reports* **526**: 93-163, https://arxiv.org/abs/1210.4997

Lüscher, 2010. "Properties and uses of the Wilson flow in lattice QCD" *JHEP* **2010(08)**: 071, https://doi.org/10.1007/JHEP08(2010)071

Magnen *et al*, 1993. "Construction of YM_4 with an Infrared Cutoff" *Commun. Math. Phys.* **155**: 325-383, https://projecteuclid.org/euclid.cmp/1104253284

Majumdar *et al*, 2004. "Glueball masses in 4d U(1) lattice gauge theory using the multi-level algorithm" *Nucl. Phys. B* **677**: 273-288, https://arxiv.org/abs/hep-lat/0309003

Makino *et al*, 2018. "Gradient flow and the Wilsonian renormalization group flow" *Prog. Theor. Exp. Phys.* **2018**: 053B02, https://academic.oup.com/ptep/article/2018/5/053B02/5021515

McGreevy, 2019. "Physics 215C: Particles and Fields" https://mcgreevy.physics.ucsd.edu/s19/2019-215C-lectures.pdf

Montvay and Münster, 1997. *Quantum Fields on a Lattice.* Cambridge University Press

Morningstar and Peardon, 2004. "Analytic Smearing of SU(3) Link Variables in Lattice QCD" *Phys. Rev. D* **69**: 054501, https://arxiv.org/abs/hep-lat/0311018v1

Morris, 2005. "Renormalizable extra-dimensional models" *JHEP* **2005(01)**: 002, https://doi.org/10.1088/1126-6708/2005/01/002

Necco, 2003. "The static quark potential and scaling behavior of SU(3) lattice Yang-Mills theory" https://arxiv.org/abs/hep-lat/0306005v1

Pérez *et al*, 2017. "Perturbative contributions to Wilson loops in twisted lattice boxes and reduced models" *JHEP* **2017(10)**: 150, https://doi.org/10.1007/JHEP10(2017)150

Peskin and Schroeder, 1995. *An Introduction to Quantum Field Theory.* Addison-Wesley

Peter, 1997. "The static potential in QCD - a full Two-Loop Calculation" *Nucl.Phys. B* **501**: 471-494, https://arxiv.org/abs/hep-ph/9702245

Smit, 2002. *Introduction to Quantum Fields on a Lattice.* Cambridge University Press

Trottier *et al*, 2002. "Perturbative expansions from Monte Carlo simulations at weak coupling: Wilson loops and the static-quark self-energy" *Phys.Rev. D* **65**: 094502, https://arxiv.org/abs/hep-lat/0111028v1

Yamanaka *et al*, 2021. "Dark matter scattering cross section and dynamics in dark Yang-Mills theory" *Physics Letters B* **813**: 136056, https://arxiv.org/abs/1910.01440

Zelobenko, 1973. *Compact Lie Groups and Their Representations.* American Mathematical Society

37 References in this series

Article **00951** (https://cphysics.org/article/00951):
"Complete Gauge Fixing"

Article **00980** (https://cphysics.org/article/00980):
"The Free Scalar Quantum Field: Vacuum State"

Article **07246** (https://cphysics.org/article/07246):
"Phase Structure of Models of Scalar Quantum Fields"

Article **10142** (https://cphysics.org/article/10142):
"Universality and Continuum Limits with Scalar Quantum Fields"

Article **18505** (https://cphysics.org/article/18505):
"Matrix Math"

Article **22212** (https://cphysics.org/article/22212):
"The Wilsonian Effective Action with Scalar Quantum Fields"

Article **23277** (https://cphysics.org/article/23277):
"Normal Ordering and Composite Operators for Free Scalar Quantum Fields"

Article **26542** (https://cphysics.org/article/26542):
"The Free Quantum Electromagnetic Field in Smooth Space"

Article **37431** (https://cphysics.org/article/37431):
"How to Think About Units"

Article **52890** (https://cphysics.org/article/52890):
"Defining Scalar Quantum Fields on a Spatial Lattice"

Article **85870** (https://cphysics.org/article/85870):
"The Force Mediated by a Scalar Quantum Field"

Article **89053** (https://cphysics.org/article/89053):
"Quantum Gauge Fields on a Spacetime Lattice"

Article **92035** (https://cphysics.org/article/92035):
"The Topology of Lie Groups: a Collection of Results"