

Asymptotic Freedom and the Continuum Limit of Yang-Mills Theory

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Abstract Article [89053](#) introduces a family of models called **Yang-Mills theories** whose only quantum field is a gauge field. The models are defined by treating spacetime as a lattice, with the understanding that we only care about what the model predicts at resolutions much coarser than the lattice scale. Many models with a nonabelian gauged group in 3- or 4-dimensional spacetime have a property called **asymptotic freedom**, which is believed to be related to the existence of a nontrivial strict continuum limit. This article uses Yang-Mills theory with gauged group $SU(N)$ to introduce that relationship. This article also clarifies some things about the small-coupling expansion (also called **perturbation theory**) in models with gauge fields.

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1 Introduction

When the gauged group G is connected and nonabelian, the models constructed in article 89053 – called (pure) **Yang-Mills theories** – are among the simplest examples in quantum field theory of models that are believed to have nontrivial continuum limits with Lorentz symmetry.^{1,2} The evidence for the existence of such a limit is related to **asymptotic freedom**, the subject of this article.

The name *asymptotic freedom* may be used for either of two closely-related mathematical phenomena. To distinguish between them, this article calls them the **bare** version and the **renormalized** version of asymptotic freedom.

This article is mostly about the bare version of asymptotic freedom. The family of lattice models considered in this article has only one continuously adjustable dimensionless parameter, which may be expressed in terms of the **bare coupling** g that will be introduced in section 13. If an appropriate low resolution³ prediction ω is chosen, then holding the value of ω fixed as the lattice spacing goes to zero relative to a fixed physical scale may force the value of g to approach zero. This is the bare version of asymptotic freedom. Calculations using a variety of methods have not found any obstructions to taking that limit when spacetime has four (or fewer) dimensions and $G = SU(N_c)$. This apparent absence of obstructions is consistent with the presumed existence of a nontrivial continuum limit.⁴ In contrast, the analogous calculation in models without asymptotic freedom indicates an obstruction to such a limit.⁵

Section 25 will introduce the renormalized version of asymptotic freedom, and section 28 will show that the bare and renormalized versions imply each other.

¹A continuum limit is called **trivial** if the model's predictions in that limit can be reproduced by a model whose action is a quadratic function of the field variables. Nontrivial continuum limits are more interesting, because they can have particles that interact with each other.

²The models in this article are initially defined on a spacetime lattice. Some progress toward a construction of Yang-Mills theory directly in continuous spacetime has also been made (Magnen *et al* (1993)).

³In this article, **low resolution** means coarse compared to the distance ϵ between neighboring points in the lattice.

⁴A full proof of the existence of such a limit is not currently known (Douglass (2004); Jaffe and Witten (2000)).

⁵Sections 31-32

2 Notation

- d = number of dimensions of spacetime
- G = the gauged group, which is always either $SU(N_c)$ or $U(1)$ in this article
- N_c = number of **colors** when the gauged group is $SU(N_c)$
- N_f = number of quark fields (**flavors**), which is zero in this article except in section 30
- N_{id} = trace of the identity matrix in the representation of G (section 4)
- ϵ = distance between neighboring points in the lattice (section 6)
- m = mass gap, so $1/(m\epsilon)$ is the correlation length in units of the lattice spacing (section 5)
- ω = a quantity whose value is held fixed as the continuum limit is being taken (section 10)
- S = the action (section 4)
- β = overall coefficient of action⁶
- g = the **bare coupling**, defined by writing $\beta \propto 1/g^2$
- g_R = a **renormalized coupling**, a special type of quantity that can be used for ω
- E = a characteristic energy associated with g_R or ω

⁶In quantum field theory, the symbol β is commonly used for two different things. One is the overall coefficient of the action. That convention that comes from the mathematical similarity between path integrals in lattice gauge theory and expectation values in statistical mechanics, where β traditionally denotes the inverse temperature. The symbol β is also commonly used for the **beta function** in the context of the renormalization group. The beta function is also important in this article, but this article doesn't use the symbol β for it (section 26).

3 Simplifying conditions

To avoid distracting complications, this article imposes two simplifying conditions on the model's definition:

- The lattice does not wrap back on itself.
- Observables are invariant under all gauge transformations, not just under gauge transformations that act trivially on the boundary.⁷

One advantage of using a lattice that wraps back on itself would be that it allows a model to have a discrete version of translation symmetry when the lattice has finite size. One disadvantage would be that setting up a small-coupling expansion is obstructed by the presence of **zero modes**, which are combinations of link variables that aren't constrained by the gauge fixing protocol and that also don't affect the value of the action.⁸ One way to eliminate zero modes is to use *twisted boundary conditions*.⁹ Another way is to overtly prune the zero modes from the definition of the path integral. The easiest way, the one used in this article, is to use a lattice that doesn't wrap back on itself at all.

⁷Article [00951](#)

⁸Article [00951](#); Heller and Karsch (1985)

⁹Trottier *et al* (2002), Pérez *et al* (2017)

4 The model on a spacetime lattice

This section reviews part of the construction that article [89053](#) describes in more detail. Let G be a compact Lie group. When spacetime is treated as a lattice, the model is constructed entirely from G -valued **link variables** $u(\ell)$, one for each directed link ℓ in the lattice. Link variables associated with oppositely-directed links are each other's inverses. This article uses the **Wilson action**

$$S[u] = \frac{\beta}{2} \sum_{\square} \left(1 - \frac{W(\square)}{N_{\text{id}}} \right), \quad (1)$$

where each **plaquette variable** $W(\square)$ is the trace of the product of the link variables around the perimeter of the plaquette \square , with the trace defined using a faithful matrix representation of G . The integer N_{id} is the trace of the matrix that represents the identity element of G . Article [89053](#) explains how these ingredients are used to define a quantum model in the path integral formulation. Schematically, the expectation value of a product of observables O_1, O_2, \dots is¹⁰

$$\langle O_1 O_2 \dots \rangle = \frac{\int [du] e^{-S[u]} O_1[u] O_2[u] \dots}{\int [du] e^{-S[u]}}, \quad (2)$$

where $O_k[u]$ is a representation of O_k in terms of link variables.

This model has only one continuously adjustable dimensionless parameter, the coefficient β in equation (1). Generalizations involving two or more such parameters can also be interesting¹¹ but will not be considered in this article.

¹⁰We could specify initial and final states by inserting the functions that represent them into the integrand (article [89053](#)). If we don't do that, then (after Wick rotation) the expectation value naturally approaches the vacuum expectation value as the length of the lattice in the time direction approaches infinity. More precisely, it approaches an average expectation value over all vacuum states, if more than one vacuum state exists. Those technicalities won't be important in this article.

¹¹Florio *et al* (2021)

5 Correlation length and mass gap

The **correlation length** characterizes how quickly correlations between simultaneous local observables fall off as a function of the distance between them when the state is the vacuum state. The correlation length is determined by the **mass gap** – the gap that separates the energies of all non-vacuum states from the energy of the vacuum state.¹² Let m denote the mass gap. If m is nonzero, then the longest-range correlations fall off like $\sim \exp(-m|x|)$ asymptotically, where $|x|$ is the distance between the observables.¹³ This says that $1/m$ is the correlation length. If m is zero (infinite correlation length), then the correlations fall off like a power of $|x|$ instead. This relationship between the mass gap and the correlation length can be deduced using the euclidean path integral formulation.^{14,15} Briefly: the euclidean “time evolution” operator is $\exp(-Ht)$, where H is the hamiltonian, so correlations between observables separated in time fall off exponentially with a minimum rate determined by m . The symmetry of euclidean spacetime then implies that the same must be true of correlations between observables separated in space. This also shows that if the correlation length $1/m$ were zero, then nothing (no particles or signals) would propagate over distances large enough for experiments to resolve. Models in which the correlation is finite or infinite in physical units can both be interesting, but models with zero correlation length are not.

When the gauged group G is nonabelian, numerical calculations give compelling evidence that Yang-Mills theory admits a continuum and infinite-volume limit with nonzero interactions and a nonzero mass gap.¹⁶ Asymptotic freedom, the subject of this article, can be studied without knowing whether the mass gap is nonzero.

¹²The name *mass gap* comes from the fact that this energy difference is equal to the mass of the lowest-mass particle (in models whose low energy states have a meaningful particle interpretation).

¹³Article [00980](#) illustrates this in the model of a free scalar field.

¹⁴Montvay and Münster (1997), equations (1.197)-(1.201)

¹⁵Article [07246](#) explains this a little more directly in the case of scalar fields.

¹⁶The strong-coupling approximation shows that the correlation length is finite in units of ϵ when β is small enough (Montvay and Münster (1997), section 3.4.5), and numerical evidence indicates that it remains finite for all β (Montvay and Münster (1997), section 3.7).

6 Continuum and infinite-volume limits

Write ϵ for the distance between neighboring lattice sites, and write K for the number of lattice sites across the width of the lattice, so that ϵK is the overall linear size of the lattice. The model is defined by treating spacetime as a finite lattice, but the only predictions that matter are the ones whose resolutions are much coarser than the lattice spacing ϵ and much finer than the overall size ϵK of space, because those scales are both artificial.

Let ρ be any unit of length such that the resolutions of realistic measurements are nonzero and finite when expressed in units of ρ . This will be called a **physical unit of length**.¹⁷ We could keep K finite as long as it is very large, but for aesthetic reasons we would prefer to take two strict limits:

- a continuum limit, which means $\epsilon/\rho \rightarrow 0$ subject to the constraint $\epsilon K/\rho > 0$,
- an infinite-volume limit, which means $\epsilon K/\rho \rightarrow \infty$.

This article reviews concepts related to taking a continuum limit. Most of this article assumes that the infinite-volume limit has already been taken.

To get a useful continuum limit, we need the correlation length to be nonzero in physical units,¹⁸ which implies that we need $m\epsilon \rightarrow 0$,¹⁹ where m is the mass gap. The quantity $m\epsilon$ is a function of the model's parameters and is independent of ρ , so reaching such a limit may require manipulating the model's parameters in addition to manipulating ρ .

¹⁷This article uses a system of units in which the speed of light and Planck's constant \hbar are both equal to 1, so if we have chosen a physical unit of length ρ , then we can use $1/\rho$ as a physical unit of energy.

¹⁸Section 5 explained why this is important.

¹⁹This implication follows from the fact that the correlation length in physical units is $1/(m\rho)$, so we need to keep $m\rho$ finite, which implies $(m\rho)(\epsilon/\rho) \rightarrow 0$ as $\epsilon/\rho \rightarrow 0$.

7 Adjusting parameters to reach a continuum limit

Suppose that we have taken an infinite-volume limit but have not yet taken a continuum limit. Article 00951 shows that if we adopt the simplifying conditions that were declared in section 3 and use the gauge-fixing protocol that will be reviewed in section 15, then the gauge-fixed form of the path integral has this property: taking β to be large suppresses all configurations in which nearby link variables deviate significantly from each other. This strongly suggests that taking $\beta \rightarrow \infty$ should make the correlation length become infinite in units of ϵ , which is equivalent to $m\epsilon \rightarrow 0$. The rest of this article assumes that this suggestion is correct.

Values of β at which $m\epsilon \rightarrow 0$ are called **critical points**. For the models with action (1), $m\epsilon$ is nonzero when β is small enough,²⁰ and it is believed to be nonzero and finite for all finite values of β when the gauged group G is nonabelian and $d \in \{3, 4\}$,²¹ so this article will only explore what happens when $\beta \rightarrow \infty$.

²⁰This is proved using the **strong coupling** (small β) **expansion**, which has a finite radius of convergence (Montvay and Münster (1997), section 3.4.5).

²¹When $d = 4$ and $G = U(1)$, good evidence exists for a **weakly first order phase transition** at $\beta = \beta_c \approx 1$ (Arnold *et al* (2001); Arnold *et al* (2002), mentioned in Majumdar *et al* (2004)). The value of $m\epsilon$ at such a phase transition would be $\ll 1$ but not zero, so it's not quite a critical point.

8 The physical length scale and continuum limits

Taking a continuum and infinite-volume limit involves two things: tuning the model's parameters to achieve $m\epsilon \rightarrow 0$ (section 7), and choosing how the physical unit ρ depends on those parameters. In a model where spacetime is already continuous, the choice of units would not have any affect on the model's physical content,²² but the way we make the units behave during the process of taking a limit can affect the physical content of the result.

As an example, suppose that the mass gap is nonzero,²³ which means that the model has an infinite-volume limit in which $m\epsilon K \rightarrow \infty$. Taking an infinite-volume limit requires $\rho/(\epsilon K) \rightarrow 0$, but we are free to choose the rate at which this quantity goes to zero: we can make ρ behave however we want, because the model on a finite lattice is independent of ρ . By choosing that rate, we can make the quantity $m\rho = (m\epsilon K)(\rho/(\epsilon K))$ approach either zero or a nonzero value, whichever we prefer.²⁴ The quantity $m\rho$ is the mass gap in physical units.

Now suppose that taking $\beta \rightarrow \infty$ gives a continuum limit.²⁵ After taking the infinite-volume limit, we still have the freedom to choose how ρ depends on β . If the mass gap m is nonzero, then we can choose $\rho(\beta)$ to make $m\rho$ do whatever we want as $\beta \rightarrow \infty$:²⁶ we can make it remain finite, or we can make it go to zero.

Even without knowing whether the mass gap is nonzero, one thing is clear: starting with one of the lattice models described in section 4, if we take an infinite-volume limit and then take $\beta \rightarrow \infty$ to get a continuum limit, then the resulting model cannot have any remaining continuously adjustable dimensionless parameters.²⁷ The qualifier *dimensionless* is essential because the model might still have a characteristic scale, like a mass gap.

²²Article [37431](#)

²³The model reviewed in section 4 is believed to have such a limit when G is nonabelian and $d \in \{3, 4\}$.

²⁴If the model doesn't have a limit with $m\epsilon K \rightarrow \infty$, then we are stuck with $m\rho \rightarrow 0$.

²⁵Section 7

²⁶Section 33 will illustrate this in a model where the dependence of $m\epsilon$ on β is known (for large β), namely the model with $G = U(1)$ and $d = 3$.

²⁷If the limits are taken in the opposite order, then the absence of continuously adjustable dimensionless parameters in the resulting model would no longer be evident (at least not to me), because K is an integer.

9 Relationship to the renormalization group(s)

If a model in infinite continuous spacetime has one or more continuously adjustable dimensionless parameters, then we can ask how (or whether) those parameters can be adjusted to emulate an overall change of scale. This defines a flow in the space of possible parameter-values. This flow is called the **renormalization group**.²⁸

Models that look the same at all scales are called **fixed points** of the renormalization group. The models traditionally studied in relativistic quantum field theory all live in neighborhoods of the **trivial** fixed point, which is the fixed point obtained by omitting all higher-than-quadratic terms in the action. Among these models, the ones with asymptotic freedom are believed to be the ones that admit nontrivial strict continuum limits. This article only considers models that live near the trivial fixed point.²⁹

After taking infinite-volume and continuum limits, some models don't have any continuously adjustable dimensionless parameters, but they also don't look the same at all scales. They have a **characteristic scale**, typically expressed in units of energy.³⁰ Example: when the gauged group G is nonabelian and $d \in \{3, 4\}$, the models described in section 4 are thought to have a nonzero mass gap,³¹ which provides a characteristic scale.^{32,33} Asymptotic freedom can be studied without knowing whether this is possible.

²⁸Article [10142](#)

²⁹When $d \geq 5$ and $G = SU(N_c)$ with $N_c \leq 4$, some evidence suggests the existence of a nontrivial fixed point – not necessarily for the Wilson action (1) (Morris (2005), section 2), but for other some choice of lattice action whose continuum limit is (4). An example of such evidence is presented in De Cesare *et al* (2021). Florio *et al* (2021) lists some additional references.

³⁰If we take the limit as this scale goes to 0 or ∞ , then the model becomes scale invariant.

³¹Section 5

³²The model's predictions also depend on other dimensionful quantities, like the momenta of the particles entering a scattering event, but those momenta are properties of the initial state, not properties of the model itself. The characteristic scale is a property of the model itself.

³³The original lattice model has one dimensionless continuously adjustable dimensionless parameter, namely β . After taking the infinite-volume and continuum limits, the resulting model has no continuously adjustable dimensionless parameters but does have a characteristic scale. This phenomenon is called **dimensional transmutation**.

10 The concept of a renormalization scheme

Suppose that we have already taken $K \rightarrow \infty$. Taking a continuum limit means choosing a physical unit of length, denoted ρ , and sending $\epsilon/\rho \rightarrow 0$.³⁴ To specify the limit unambiguously, we need to specify a relationship between β and ϵ/ρ . We can do that by choosing a function $\omega(\beta, \epsilon/\rho)$ that depends on both β and ϵ/ρ and requiring that ω remain constant as $\epsilon/\rho \rightarrow 0$. This is called choosing a **renormalization scheme**.³⁵ It implicitly specifies β as a function of ϵ/ρ , which will be denoted $\beta(\epsilon/\rho)$ in this article.³⁶

Physically, the function $\omega(\beta, \epsilon/\rho)$ should be related to a family of meaningful low resolution predictions that is parameterized by the physical scale ρ in the continuum limit.³⁷ Section 11 will list a few examples of renormalization schemes that are used when $d = 4$ and $G = SU(N_c)$, and section 21 will highlight a shared feature of those schemes.

³⁴Section 6

³⁵The word *renormalization* is used for a few related-but-different things. The way it's used in this section could be called *nonperturbative renormalization*, because its definition doesn't refer to any small-parameter expansion. In contrast, *perturbative renormalization* is a way of constructing useful small-parameter expansions (section 24).

³⁶The function denoted $\beta(\epsilon/\rho)$ in this article is distinct from what is traditionally called the *beta function* in quantum field theory (section 26).

³⁷The relationship between those physical predictions and ω may be indirect, and this freedom can be used to help make calculations easier.

11 Various renormalization schemes

We can specify the function $\beta(\epsilon/\rho)$ by specifying a low resolution quantity ω whose value is kept fixed as $\epsilon/\rho \rightarrow 0$. When $G = SU(N_c)$ and $d = 4$, quantities that have been used in the literature for that purpose include:

- quantities expressed in terms of n -point functions of the gauge field,³⁸
- quantities expressed in terms of the expectation value of a Wilson loop,³⁹
- a smoothed version of $\text{trace}(F_{ab}^2(x))$, using a recipe called the **gradient flow** to do the smoothing.^{40,41}

Section 21 will highlight a feature shared by all these approaches.

Another way to specify the function $\beta(\epsilon/\rho)$ is to find a modified action on a coarser lattice that gives the same low resolution predictions as the original action on the finer lattice.⁴² That approach is more difficult and is not normally used in practice, at least in models with gauge fields.⁴³

³⁸Sections 16-19 will explain how these functions are defined, and section 23 will describe this type of scheme in more detail.

³⁹Section 22 will review this scheme.

⁴⁰Lüscher (2010), equation (1.4) and the surrounding text, and equations (2.1), (2.14), and (3.1)

⁴¹Makino *et al* (2018) explains how this approach relates to the zooming-in/out idea used in article 10142. The smoothing effect is related to the one described more explicitly in Morningstar and Peardon (2004). Fodor *et al* (2012) describes a variation of the method, using finite volume to eliminate complications due to zero modes.

⁴²For a family of models involving only scalar fields, article 22212 uses a similar idea that works by integrating out high-momentum modes in the path integral.

⁴³Balaban (1987) explains how to implement it in Yang-Mills theory with the action (1). Theorem 2 in Balaban (1987) (for $d = 4$, and the text above it for $d = 3$) gives the form of upper and lower bounds on the (inverse of the) renormalized coupling defined using this procedure.

12 Preview of results

In a continuum limit, the correlation length must become infinite in units of the lattice spacing.⁴⁴ Section 7 established that this requirement can be satisfied by taking $\beta \rightarrow \infty$. For $G = SU(N_c)$ this appears to be the only way to satisfy that requirement, so for a given renormalization scheme, the key question is whether it makes $\beta \rightarrow \infty$. For $G = SU(N_c)$, the result of applying any of the standard renormalization schemes depends on d :

- For $d \leq 4$, it drives β toward ∞ as $\epsilon/\rho \rightarrow 0$, as desired.
- For $d \geq 5$, it drives β toward zero.⁴⁵

The $d \neq 4$ cases will be deduced in section 29. The $d = 4$ case requires additional calculations, the results of which will be reviewed in section 30.

⁴⁴Section 6

⁴⁵For $d \geq 5$, instead of imposing any of the standard renormalization schemes, we could independently send both $\epsilon/\rho \rightarrow 0$ and $\beta \rightarrow \infty$. That would give a continuum limit with nonzero correlation length, but the resulting model would be trivial (no interactions).

13 The bare coupling

The overall coefficient β is the only continuously adjustable parameter in the action (1). Calculations are often done by writing β in terms of the bare coupling g that was mentioned in section 1 and then absorbing a factor of g into the gauge field so that an expansion in powers of g is essentially an expansion in powers of the gauge field.⁴⁶ This section introduces the bare coupling g , using the small- g expansion of the action itself to motivate the details of the definition.

Let G be a (faithful matrix representation of a) compact Lie group. Let T_1, T_2, \dots be a complete set of generators for the Lie algebra, chosen to satisfy

$$\text{trace}(T_j T_k) = -\nu \delta_{jk}$$

for some $\nu > 0$.⁴⁷ Define the bare coupling $g > 0$ and the gauge field $A_a(x)$ by writing the coefficient β and the link variables $u(\ell)$ as

$$\beta = \frac{2N_{\text{id}}\epsilon^{d-4}}{\nu g^2} \quad u(\ell) = \exp(g\epsilon A_a(x)), \quad (3)$$

where N_{id} is the trace of the identity matrix, and the endpoints of the link ℓ are x and $x + e_a$, using the subscript a to specify which dimension is parallel to the link. Each component $A_a(x)$ of the gauge field is a linear combination of the generators T_k . The definition of g in (3) is motivated by the fact that non-quadratic terms in the action are multiplied by extra powers of g compared to the quadratic term.⁴⁸

$$S = \frac{-1}{4\nu} \int d^d x \sum_{a,b} \text{trace}(\partial_a A_b - \partial_b A_a)^2 + O(g)O(A^3) \quad (4)$$

where ∂ is a lattice version of the derivative, which includes a factor of ϵ in the denominator, and $\int d^d x \dots$ is a lattice version of the integral over spacetime, which includes a factor of ϵ^d in the numerator.

⁴⁶Section 20

⁴⁷The conventions $\nu = 1$ and $\nu = 1/2$ are typically used when G is abelian and nonabelian, respectively.

⁴⁸article [89053](#)

14 Asymptotic freedom and nontriviality

Knowing that a model has asymptotic freedom does not automatically prove that it has a nontrivial continuum limit, but among models that live in a neighborhood of the trivial fixed point, those that have asymptotic freedom are thought to be the ones that have nontrivial continuum limits, partly because these are the models that don't show evidence of any obstruction to such a limit (section 32).

That might seem counterintuitive, because (the bare version of) asymptotic freedom means that $g \rightarrow 0$ in the continuum limit, and the continuum limit would be trivial if the non-quadratic terms in the action (4) were absent. However, some effects of the non-quadratic terms in (4) – the terms that are multiplied by a positive power of g – may approach nonzero values even though the non-quadratic terms themselves approach zero.⁴⁹ The fact that $g \rightarrow 0$ in the continuum limit does not rule out the possibility that the continuum limit is nontrivial.

⁴⁹Analogy: even if $x \rightarrow 0$, a quantity of the form $x f(x)$ may remain nonzero, like when $f(x) = 1/x$.

15 Gauge-fixed expectation values

Expanding vacuum expectation values in powers of $1/\beta$ can be a useful tool for studying the limit $\beta \rightarrow \infty$. This section reviews an identity that can be used to define an expansion in powers of $1/\beta$.

A **maximal tree** may be roughly defined as a collection of (undirected) links that doesn't include any loops but that would include a loop if any other link were added to the collection.⁵⁰ Let T be the set of all directed links corresponding to the undirected links in a maximal tree. Article 00951 shows that if $I[u]$ is any gauge invariant function of the link variables, then

$$\frac{\int [du] e^{-S[u]} I[u] \cdots}{\int [du] e^{-S[u]}} = \frac{\int [du]' e^{-S[u']} I[u'] \cdots}{\int [du]' e^{-S[u']}} \quad (5)$$

where $[du]'$ indicates integration only over the link variables that are not associated with links in T , and

$$u'(\ell) = \begin{cases} u(\ell) & \text{if } \ell \notin T \\ 1 & \text{if } \ell \in T. \end{cases}$$

This is a relatively simple example of **gauge fixing**, one in which the link variables associated with links in T are all constrained to be equal to 1.⁵¹ The left-hand side of (5) will be called the **unconstrained** form, and the right-hand side will be called the **gauge-fixed** form.⁵²

⁵⁰Article 00951

⁵¹Baaquie (1977) compares two different gauge-fixing schemes in lattice Yang-Mills theory. One is a maximal-tree scheme as described here, and the other is a more symmetric scheme that makes calculations easier to manage. The **temporal gauge** and the **axial gauge** are close relatives of the type of gauge-fixing used here (article 00951).

⁵²Analyses that appear to define correlation functions of non-gauge-invariant quantities are often implicitly using an identity like (5): they define the correlation function using a particular gauge-fixing scheme, typically without ever specifying the corresponding gauge-invariant quantity. Texts often emphasize that gauge fixing is a prerequisite for using perturbation theory, and the result derived in section 17 shows that it can also be a prerequisite for defining such correlation functions nonperturbatively. Example: Henty and Parrinello (1995) use the Landau gauge to define n -point correlation functions of the lattice gauge field, even though their computational method is nonperturbative.

16 n -point correlation functions

In much of the literature about Yang-Mills theory, the quantity that is held fixed as $\beta \rightarrow \infty$ is defined using a 3- or 4-point correlation function of the variables A_a that were introduced through equation (3).⁵³ This section takes a first step toward defining those correlation functions. Section 19 will give a complete definition.

The lattice model is constructed in terms of G -valued link variables $u(\ell)$. As explained in section 20, the Lie-algebra-valued quantities $A_a(x)$ are not uniquely determined by the link variables, but if we define⁵⁴

$$\hat{A}_a(x) \equiv \hat{A}(\ell) \equiv \frac{u(\ell) - u^{-1}(\ell)}{2g\epsilon} \quad (6)$$

for the link ℓ with endpoints x and $x + e_a$, then equation (3) says that $\hat{A}_a(x)$ is equal to $A_a(x)$ to lowest order in the small- g expansion.

Correlation functions of the quantities $\hat{A}(\ell)$ are unambiguously defined in the lattice model, independently of the small- g approximation. Those correlation functions are not quite what we want, though: section 17 will show that if $\ell_1, \ell_2, \dots, \ell_n$ is a list of links that don't share any endpoints with each other,⁵⁵ then the corresponding n -point correlation function – the vacuum expectation value of the product $\hat{A}(\ell_1)\hat{A}(\ell_2)\cdots\hat{A}(\ell_n)$ – is identically zero. This is a consequence of the fact that the action is invariant under gauge transformations but $\hat{A}(\ell)$ is not.

If those correlation functions are zero, then what are textbooks really calculating when they claim to be calculating a nonzero 3- or 4-point function in Yang-Mills theory? Section 19 will answer that question.

⁵³Section 23

⁵⁴Equation (21) in Giusti *et al* (2001) takes the traceless part of (6). This article doesn't, because it's not appropriate when $G = U(1)$, and because it isn't strictly needed even when $G = SU(3)$: the non-traceless part (like the θ^3 term in $e^{i\theta} - e^{-i\theta}$ with $\theta \propto \text{diag}(2, -1, -1)$) is of higher order in g .

⁵⁵If they did, then the continuum limit of the correlation function would involve the product of two A s at the same point in spacetime. Such products require special treatment even in the simplest models (article [23277](#)).

17 A lemma about correlation functions

This section derives a result that was used in section 16.

Let Γ be a set of distinct directed links, and suppose that at least one of the two endpoints of at least one of the links in Γ is not shared by any of the other links in Γ . Let $u(\ell_1)u(\ell_2)\cdots u(\ell_n)$ be a product of link variables, in any order, with $\{\ell_1, \ell_2, \dots, \ell_n\} = \Gamma$. This section shows that the expectation value of that product is zero.

Write the path integral as

$$\langle u(\ell_1)u(\ell_2)\cdots u(\ell_n) \rangle \propto \int \prod_{k=1}^n (du(\ell_k)u(\ell_k)) f(u(\ell_1), \dots, u(\ell_n))$$

with

$$f(u(\ell_1), \dots, u(\ell_n)) \equiv \int \left(\prod_{\ell \notin \Gamma} du(\ell) \right) e^{-S[u]}.$$

The quantity $f(u(\ell_1), \dots, u(\ell_n))$ is gauge invariant and depends only on the link variables $u(\ell_1), \dots, u(\ell_n)$. We assumed that one of those link variables, say ℓ_1 , has an endpoint that is not shared by any of the others, so gauge invariance implies that the function f must be independent of $u(\ell_1)$. That gives

$$\langle u(\ell_1)u(\ell_2)\cdots u(\ell_n) \rangle \propto \left(\int \prod_{k=2}^n (du(\ell_k)u(\ell_k)) f \right) \left(\int du(\ell_1) u(\ell_1) \right).$$

The second factor is zero because the Haar integral over any compact Lie group has the property $\int dg g = 0$.⁵⁶

⁵⁶This is a special case of theorem 5.12 in Bröcker and tom Dieck (1985).

18 Another lemma about correlation functions

This section derives a result that will be used in section 19.

Let Γ be a set of directed links that forms a closed loop:

$$\Gamma = \{(x_1, x_2), (x_2, x_3), \dots, (x_n, x_{n+1})\}$$

with $x_{n+1} = x_1$. The result that was derived in section 17 doesn't apply to this case, because endpoint of every link in Γ is shared by two links in Γ . This section shows that if G is an irreducible matrix representation of a compact Lie group, like the defining representation of $U(1)$ or $SU(N)$, then the expectation value of the product

$$p \equiv u(x_1, x_2)u(x_2, x_3) \cdots u(x_n, x_1)$$

is proportional to the identity matrix, which implies that the proportionality factor is $\langle \text{trace}(p) \rangle / N_{\text{id}}$, where N_{id} is the trace of the identity matrix.

That result can be derived by combining these two facts:

- The effect of a gauge transformation on p is $p \mapsto hph^{-1}$ where h is an element of G associated with the point x_1 . This implies that the effect of the same gauge transformation on the expectation value $\langle p \rangle$ is $\langle p \rangle \mapsto h\langle p \rangle h^{-1}$.
- Everything else in the integrand of the path integral is invariant under gauge transformations, and so is the Haar measure, so the effect of this gauge transformation may be absorbed into a redefinition of the integration variables $u(x_1, x_2)$ and $u(x_n, x_1)$. This shows that $\langle p \rangle$ is invariant under gauge transformations.

Combining these two results gives $h\langle p \rangle h^{-1} = \langle p \rangle$ for all $h \in G$. If G is an irreducible representation of a compact Lie group, then this implies⁵⁷ that $\langle p \rangle$ is proportional to the identity matrix, as claimed.

⁵⁷This is a consequence of **Schur's lemma** (Zelobenko (1973), section 20, page 56).

19 n -point correlation functions with gauge-fixing

Section 17 showed that a naïve definition of the correlation function of n factors of the gauge field A is identically zero if the factors are separated from each other in spacetime. When textbooks calculate a nonzero result for such correlation function, they're implicitly using an identity like (5), where the right-hand side looks like a correlation function of n factors of the gauge field even though the function $I[u]$ on the left-hand side is invariant under gauge transformations. This section explains how that works, using the gauge-fixing protocol that was reviewed in section 15.

As an example, suppose that the function $I[u]$ on the left-hand side of the identity (5) is a **Wilson loop**, the trace⁵⁸ of the product of link variables around some closed path in the lattice. On the right-hand side of (5), some of the link variables in the given Wilson loop might have been replaced by 1. In that case, the right-hand side doesn't look like the expectation value of a Wilson loop, and it doesn't look gauge invariant. If most of the factors in the given Wilson loop are associated with links in the maximal tree T , then the function $I[u']$ on the right-hand side might be a product of only a few link variables whose links don't share any endpoints with each other.

Conversely, suppose that the function $I[u']$ on the right-hand side of (5) is a product of n link variables $u(x_1, y_1), u(x_2, y_2), \dots, u(x_n, y_n)$ whose links (x_k, y_k) don't share any endpoints with each other and don't belong to the tree T . The premise that the tree T is *maximal* means that it includes a (unique) path connecting any two given points in the lattice, so we can promote our product of n link variables to a closed loop C by inserting factors along the paths in T that connect y_1 to x_2 , connect y_2 to x_3 , and so on, finally connecting y_n back to x_1 . That doesn't change the product, because all the factors we inserted are equal to 1. This shows that the gauge-fixed expectation value of the product of n separated link variables (the left-hand side of (5)) is secretly the full expectation value of a product of link variables around a closed loop (the right-hand side of (5)). Section 18 showed that

⁵⁸The trace is necessary for making the Wilson loop gauge-invariant when G is nonabelian, but the result would be essentially the same even if we defined $I[u]$ without the trace, because the path integral automatically projects the result to its trace whenever G is an irreducible representation of a compact Lie group (section 18).

taking the trace doesn't change the result except for an overall factor of N_{id} , so the result is gauge invariant.

If the function $I[u']$ on the right-hand side of (5) is a product of n factors of the quantity \hat{A} defined in (6), then

$$\langle \hat{A}(x_1, y_1) \hat{A}(x_2, y_2) \cdots \hat{A}(x_n, y_n) \rangle \propto \sum_{s_1, s_2, \dots, s_n \in \{\pm 1\}} (s_1 s_2 \cdots s_n) \langle u^{s_1}(x_1, y_1) u^{s_2}(x_2, y_2) \cdots u^{s_n}(x_n, y_n) \rangle, \quad (7)$$

so the corresponding function $I[u]$ on the left-hand side is a linear combination of 2^n different Wilson loops⁵⁹ (or 2^{n-1} if we don't separately count loops that circulate in opposite directions but are otherwise the same).

Altogether, this shows that the correlation function of n widely-separated factors of the gauge field can be nonzero when it's defined using a particular gauge-fixing scheme. This section considered only one gauge-fixing scheme, but a similar principle applies to all of them.

⁵⁹The relationship $u^{-1}(\ell) = u(\tilde{\ell})$, where $\tilde{\ell}$ and ℓ are oppositely-directed versions of the same undirected link, implies that a different Wilson loop must be used for each term on the right-hand side of (7).

20 From large β to small coupling

We can write each G -valued link variable as

$$u(\ell) = e^{\theta(\ell)}, \quad (8)$$

where $\theta(\ell)$ is an element of the Lie algebra of G .⁶⁰ The study of asymptotic freedom uses a **small-coupling expansion**,⁶¹ also called **perturbation theory**, which amounts to expanding the link variables (8) in powers of $\theta(\ell)$ everywhere in the path integral (2). This approximation is used frequently, but I have not found a complete explanation of when and why it works. The next paragraph gives a partial explanation, and then the rest of this section clarifies why the explanation is incomplete.

Article 00951 explains that if the gauge-fixing scheme reviewed in section 15 is used, then configurations in which the link variables u differ significantly from 1 are strongly suppressed when β is large.⁶² The condition $\theta(\ell) \approx 0$ implies $u(\ell) \approx 1$, so configurations with $\theta(\ell) \approx 0$ are still important when β is large. This gives a partial justification for using a small- θ approximation.

That partial justification isn't quite enough, though, because the condition $u(\ell) \approx 1$ does not imply $\theta(\ell) \approx 0$. The rest of this section uses the cases $G = U(1)$ and $G = SU(2)$ to emphasize that fact.

When $G = U(1)$, we can write $u(\ell) = e^{i\phi(\ell)}$ for a real-valued variable $\phi(\ell)$, and then the condition $u(\ell) \approx 1$ implies $\phi(\ell)$ is limited to neighborhoods of $2\pi n(\ell)$ for all integers $n(\ell)$, not just $n(\ell) = 0$. A careful formulation of the large- β approximation should account for all these neighborhoods. The **Villain model** does this by⁶³ replacing the action (1) with an action quadratic in the quantity $\phi(\ell) - 2\pi n(\ell)$

⁶⁰This article assumes that G is defined using a faithful matrix representation, so $\theta(\ell)$ is a matrix of the same size as $u(\ell)$ and the definition of $e^{\theta(\ell)}$ is straightforward (article 18505).

⁶¹Calculations in models with more than one small coupling parameter usually involve expanding in powers of each of those parameters. Questions about how to treat different powers of different parameters consistently are avoided by using the **loop expansion** (Fradkin (2022), section 13.1).

⁶²More generally, a large- β approximation is a saddle-point approximation: configurations that don't minimize the action are suppressed when β is large.

⁶³Janke and Kleinert (1986)

and replacing the integral over each link variable $u(\ell)$ in (2) with an integral over $\phi(\ell)$ and a sum over $n(\ell)$. When β is large, the path integral is dominated by values of $\phi(\ell)$ in neighborhoods of $2\pi n(\ell)$ for all integers $n(\ell)$. This is true both for the Villain model and for the original $G = U(1)$ model with action (1), so the predictions of these two models are expected to become indistinguishable from each other when β is large enough.⁶⁴ When $d = 3$, using a small- θ expansion without the sum over $n(\ell)$ can lead to qualitatively different predictions when β is large but finite.⁶⁵ This demonstrates that the small- θ approximation is not quite automatically implied by the large- β approximation.

Now consider the case $G = SU(2)$. In this case, the condition $u(\ell) = 1$ is equivalent to the condition $\theta(\ell) = 2\pi i n(\ell) U(\ell) D U^{-1}(\ell)$, where $n(\ell)$ is an arbitrary integer, D is a traceless diagonal matrix with determinant -1 , and $U(\ell)$ is an arbitrary unitary matrix.⁶⁶ To construct a Villain-like model for $G = SU(2)$, the sum over the integer-valued variables $n(\ell)$ would need to be augmented by integrals over the unitary matrices $U(\ell)$. The resulting path integral might not be much easier to handle (either intuitively or mathematically) than the original path integral over the $SU(2)$ -valued link variables. The difficulty is even greater for other compact Lie groups G with more complicated topologies.⁶⁷ This article will cope with that difficulty the same way published sources typically cope with it – namely by ignoring it, pretending that the large- β approximation implies the small- θ approximation.

⁶⁴Janke and Kleinert (1986)

⁶⁵The continuum limit (which involves taking $\beta \rightarrow \infty$) can be taken in such a way that the small- θ approximation gives the right answer, but when $d = 3$ the continuum limit can also be taken a different way that has qualitatively different properties. Section 33 reviews both of these limits.

⁶⁶Topologically, this set of values of $\theta(\ell)$ is a series of concentric 2-spheres in \mathbb{R}^3 with radii proportional to $|n(\ell)|$. The case $G = U(1)$ can be described in a similar way, because the quantities $2\pi n$ form a series of concentric 0-spheres (a 0-sphere is a pair of points) in \mathbb{R}^1 with radii proportional to $|n|$.

⁶⁷Article [92035](#)

21 A feature shared by various schemes

Section 11 listed examples of quantities ω that can be held fixed as $\epsilon/\rho \rightarrow 0$, which determines how β behaves in that limit. The choice of which quantity to hold fixed is called a **renormalization scheme**. Section 22 will review one example of a renormalization scheme in more detail. This section highlights a feature shared by all of the most commonly-used renormalization schemes when $d = 4$.

In practice, the quantity ω to be held fixed is calculated using a small-coupling expansion,⁶⁸ whose relationship to the large- β approximation was partially explained in section 20. This leads to an equation for the ϵ/ρ -dependence of β in the small-coupling expansion.

Suppose that the gauge has been fixed⁶⁹ so that the small-coupling expansion is well-defined.⁷⁰ In the renormalization schemes that are used most often when $d = 4$, the quantity that was denoted ω in section 11 is denoted g_R^2 , and its small-coupling expansion has the form^{71,72}

$$g_R^2 = g^2 + c_4 g^4 + c_6 g^6 + \cdots . \quad (9)$$

The quantity g_R is called a **renormalized coupling**, and the renormalization scheme is defined by holding g_R fixed as $\epsilon/\rho \rightarrow 0$.

⁶⁸This is why section 11 didn't list any renormalization scheme based on holding a two-particle scattering amplitude fixed. Even though Yang-Mills theory does have particles (called **glueballs**), their existence is not visible in a small-coupling expansion. Yamanaka *et al* (2021) studies glueball scattering using numerical methods.

⁶⁹Section 15

⁷⁰The series may be well-defined even if it doesn't converge. Small-coupling expansions are typically asymptotic (non-convergent) expansions.

⁷¹Creutz (1981), equation (2.23); Montvay and Münster (1997), equation (3.257)

⁷²The coefficients c_k with $k \geq 4$ may grow without bound in the limit $\epsilon/\rho \rightarrow 0$. That's not necessarily a problem. Depending on their signs, it could just mean that g^2 is forced to approach zero in the limit $\epsilon/\rho \rightarrow 0$ if g_R^2 is held fixed (footnote 88 in section 24).

22 Example of a renormalization scheme

In one of the renormalization schemes that was listed in section 11, the quantity g_R in equation (9) is defined using a Wilson loop, specifically a rectangular Wilson loop that is arbitrarily long in the “time” direction⁷³ and with a variable width r in the space direction.⁷⁴ If $\langle W \rangle$ denotes the expectation value of the Wilson loop and t denotes the “duration” of the rectangle, then the quantity

$$V(r) \equiv - \lim_{t \rightarrow \infty} \frac{\log \langle W \rangle}{t}$$

can be interpreted as the interaction potential between two quarks in the limit of infinite quark mass, called the **static potential**.^{75,76} Taking a Fourier transform with respect to r and multiplying by an appropriate power of the Fourier-conjugate variable E gives a quantity g_R of the form (9).⁷⁷ This g_R is invariant under gauge transformations, so gauge-fixing is not a prerequisite for defining it,⁷⁸ but gauge-fixing is still a prerequisite for using a small- g expansion to calculate g_R as previewed in section 21.⁷⁹

⁷³“Time” is in quotation marks because the calculations are normally done in the context of the euclidean path integral, which does not distinguish between time and space.

⁷⁴Equation (2.7) in Celmaster and Kovacs (1984) shows how the expectation value of this Wilson loop depends on the “duration” and width of the rectangle to order $O(g^2)$ for $G = SU(N_c)$ and $d = 4$.

⁷⁵Necco (2003), equation (2.10); Bali and Boyle (2002), equation (27)

⁷⁶Article [85870](#) uses a similar idea to study the force between two charge objects (in the limit of infinite mass) mediated by a massless scalar quantum field.

⁷⁷Necco (2003), equations (2.14)-(2.15)

⁷⁸This contrasts with schemes that use an n -point correlation function for ω : gauge-fixing is a prerequisite for defining those correlation functions (section 19).

⁷⁹Section 5.1 in Smit (2002) summarizes the calculation, and Bali and Boyle (2002) lays it out in detail. (The latter source includes an arbitrary number n_f of quark fields, but we can set $n_f = 0$ to get pure Yang-Mills theory.)

23 Another example of a renormalization scheme

In another one of the renormalization schemes that was listed in section 11, the quantity g_R in equation (9) is expressed in terms of n -point functions of the gauge field.^{80,81} This is similar to what is typically done in models of scalar fields,⁸² and it is also standard in gauge theory, at least in quantum field theory texts that don't use a lattice. It can also be used in lattice gauge theory.⁸³

Different renormalization schemes can lead to different behaviors of $\beta(\epsilon/\rho)$. In particular, when applied to the model with $G = U(1)$, keeping the connected part of a four-point function (which would be zero if the action were quadratic) fixed as $\epsilon/\rho \rightarrow 0$ would make $\beta \rightarrow 0$, whereas defining g_R using the Wilson-loop recipe that was described in section 22 would make β approach a finite nonzero constant.⁸⁴ In contrast, for $G = SU(N_c)$, those two different renormalization schemes both lead to the same result.

⁸⁰Sections 16-19 explained how these functions are defined.

⁸¹Instead of using the full n -point function, this type of renormalization scheme typically uses the **one-particle irreducible (1PI)** part, because this would be zero if interactions were absent. Beware that the use of the word *particle* in this name comes from perturbation theory. It doesn't necessarily refer to physical particles.

⁸²Article [22212](#)

⁸³Hasenfratz and Hasenfratz (1980) (which corrects errors in Dashen and Gross (1981)) use this approach in lattice Yang-Mills theory, and Montvay and Münster (1997) reviews the results in equations (3.256)-(3.261).

⁸⁴Smit (2002), equations (5.19)-(5.21)

24 Perturbative renormalization

Truncating the expansion (9) at a finite power of g is not necessarily a good approximation, because the coefficients c_k in the expansion may have large magnitudes.^{85,86} so the coefficient of the g^4 term diverges when $\epsilon/\rho \rightarrow 0$. Even though the small- g expansion may not be directly useful as an approximation, it is still indirectly useful:

- It can be used to infer something about the behavior of $\beta \sim \epsilon^{d-4}/g^2$ as $\epsilon/\rho \rightarrow 0$,⁸⁷ which then tells us something about the nature of the continuum limit.⁸⁸ This is the subject of sections 26 and 29-30.
- An expansion in powers of g can be rearranged into an expansion in powers of g_R , and truncating this expansion at a finite power of g_R can give a good approximation of some physically important quantities in the continuum limit even if truncating the original g -expansion was not a good approximation. This trick is called **perturbative renormalization**. Perturbative renormalization is the context for the concept of a *running coupling*, which will be introduced in section 25.

⁸⁵Lepage and Mackenzie (1993), section 2.1

⁸⁶Example: $c_4 \sim \log(\epsilon/\rho)$ when $d = 4$ and $G = SU(N_c)$. This is implied by the results that will be reviewed in section 30, which say that the quantity $\gamma_4 \equiv -\epsilon dc_4/d\epsilon$ in equation (11) is independent of ϵ/ρ .

⁸⁷The desired behavior is $\beta \rightarrow \infty$ (section 7).

⁸⁸For a contrived example, consider the finite series $g_R^2 = g^2 - g^4 \log(\epsilon/\rho) + g^6 \log^2(\epsilon/\rho)$. Holding this quantity fixed as $\epsilon/\rho \rightarrow 0$ forces $g \rightarrow 0$, which gives $\beta \rightarrow \infty$ if $d \leq 4$. This remains true if the g^6 term is ignored, even though ignoring the g^6 term is not a good approximation when $|g^2 \log(\epsilon/\rho)| \gg 1$.

25 Asymptotic freedom: renormalized version

The preceding sections focused on what section 1 called the bare version of asymptotic freedom. In most of the physics literature, **asymptotic freedom** refers to what section 1 called the renormalized version of asymptotic freedom. This section introduces the renormalized version.

Exact calculations are rarely feasible in quantum field theory, so calculations are typically done by working out the first few terms in an expansion in powers of some small parameter. We can take this small parameter to be the quantity g_R that was introduced in section 21, because the value of that quantity is kept fixed in the continuum limit.

Given a one-parameter family of predictions $P(g_R, E)$ parameterized by the total energy E of the initial state, we can define a function $g_R(E)$ so that

$$P_{\text{tree}}(g_R(E), E) = P(g_R, E),$$

where $P_{\text{tree}}(\cdot, \cdot)$ is the lowest-order term in the small- g_R expansion of the function $P(\cdot, \cdot)$. By choosing an energy E_0 and expanding things in powers of $g_R(E_0)$, the first few terms in the expansion can be a good approximation for physical processes whose energy is close to E_0 , where $g_R(E)$ is small.⁸⁹ The function $g_R(E)$ is called a **running coupling**. In Yang-Mills theory, the function $g_R(E)$ approaches zero as $E \rightarrow \infty$, so this small-coupling expansion becomes an increasingly good approximation at higher energies. This is the renormalized version of asymptotic freedom.⁹⁰

In practice, we don't know how to calculate $P(g_R, E)$ or $g_R(E)$ exactly, but we can make progress by using perturbative renormalization:⁹¹ we can calculate everything as a formal expansion in powers of the bare coupling g , and then we can invert the relationship (9) to express the bare coupling in terms of $g_R(E_0)$ so that everything ends up being expanded in powers of $g_R(E_0)$ instead.

⁸⁹We can use $\rho = 1/E_0$ as the physical length scale in section 10.

⁹⁰Even though it's defined in the context of a small-coupling expansion, the renormalized version of asymptotic freedom can be checked using numerical calculations (Creutz (1981)).

⁹¹Section 24

26 The beta function

In a renormalization scheme of the type described in section 21, a quantity of the form (9) is held fixed as $\epsilon/\rho \rightarrow 0$. This condition may be expressed as

$$D_\epsilon g_R^2 = 0, \quad (10)$$

using the abbreviation $D_\epsilon \equiv \epsilon d/d\epsilon$ for the dimensionless derivative. Equations (9) and (10) together define how g^2 depends on ϵ , which in turn defines how the quantity β in equation (3) depends on ϵ .

Equation (10) is implicitly a differential equation for the function $g^2(\epsilon)$. To make it explicit, use equation (9) to get

$$D_\epsilon g_R^2 = (1 + 2c_4 g^2 + 3c_6 g^4 + \dots) D_\epsilon g^2 + (D_\epsilon c_4) g^4 + (D_\epsilon c_6) g^6 + \dots$$

and then use this in (10) to get

$$\frac{1}{2} D_\epsilon g^2 = \gamma_4 g^4 + \gamma_6 g^6 + \dots \quad (11)$$

with g -independent coefficients γ_k . The abbreviation

$$\gamma \equiv \frac{1}{2} D_\epsilon g^2 \quad (12)$$

will be used for the left-hand side of (11). The quantity (12) is called the **beta function** and is often denoted $\beta(g)$. This article is using the symbol β for a different purpose, namely for the overall coefficient of the action (1). Holding g_R^2 fixed as $\epsilon/\rho \rightarrow 0$ requires treating that parameter β as a function of ϵ/ρ , and this article uses the notation $\beta(\epsilon/\rho)$ for that function, not for the quantity (12). The quantity (12) will also be important, though: it will be used to access the sign of the slope of the function $\beta(\epsilon/\rho)$. This sign can be used to diagnose asymptotic freedom.

27 Scheme-independence of the first coefficients

Equation (9) implies that the functions $g(\epsilon)$ and $\tilde{g}(\epsilon)$ defined by two different renormalization schemes in that family are related to each other by

$$\tilde{g}^2(\epsilon) = g^2(\epsilon) + cg^4(\epsilon) + O(g^6). \quad (13)$$

This section shows that the coefficients of the first two terms in the small-coupling expansion of the quantity

$$\gamma \equiv \epsilon \frac{d}{d\epsilon} g^2 \quad (14)$$

are the same for all such renormalization schemes. To prove this, write

$$\gamma = \gamma_4 g^4 + \gamma_6 g^6 + O(g^8) \quad (15)$$

and use

$$\begin{aligned} \epsilon \frac{d}{d\epsilon} \tilde{g}^2 &= \epsilon \frac{d}{d\epsilon} (g^2 + cg^4 + O(g^6)) && \text{(equation (13))} \\ &= \gamma \times (1 + 2cg^2 + O(g^4)) && \text{(equation (14))} \\ &= (\gamma_4 g^4 + \gamma_6 g^6 + O(g^8)) \times (1 + 2cg^2 + O(g^4)) && \text{(equation (15))} \\ &= \gamma_4 g^4 + \gamma_6 g^6 + 2c\gamma_4 g^6 + O(g^8) \\ &= \gamma_4 \tilde{g}^4 + \gamma_6 \tilde{g}^6 + O(\tilde{g}^8). && \text{(equation (13))} \end{aligned}$$

Comparing the last line to equation (13) shows that the first two coefficients in (11) are the same for two different renormalization schemes defined by fixing different quantities of the form (9).⁹²

⁹²The text below equation (4.1) in Fodor *et al* (2012) mentions a more general family of renormalization schemes in which the quantity held fixed is not limited to the form (9). Among renormalization schemes in that more general family, only the first coefficient is scheme-independent.

28 Asymptotic freedom: bare and renormalized

The preceding sections focused on what section 1 called the bare version of asymptotic freedom, which is about what happens to the bare coupling g when an appropriate prediction is held fixed as the lattice spacing ϵ goes to zero compared to a given physical scale. Expanding to only a finite power of the bare coupling might not always give a good approximation,⁹³ but it can lead to a good approximation when rearranged to be an expansion in powers of a **renormalized coupling** g_R that is more closely related to (or equal to) the given prediction and that is related to the bare coupling by a series of the form (9). The coefficients in that series are functions of the lattice spacing. For $d = 4$, the quantities g and g_R are dimensionless, so the coefficients depend on the lattice spacing ϵ only through the dimensionless combination ϵ/ρ where ρ is a given physical length scale.⁹⁴ In practice, the scale ρ is usually taken to be the inverse of an energy scale E that characterizes the physical process used to define g_R ,⁹⁵ so the dimensionless combination is ϵE . The renormalized version of asymptotic freedom is about what happens to g_R as the characteristic energy E goes to infinity, or equivalently as the characteristic length $\rho = 1/E$ goes to zero.

At least to first order in the small-coupling expansion when $d = 4$, the bare and renormalized versions of asymptotic freedom imply each other. To deduce this, start with

$$g_R^2 = g^2 + f(\epsilon E)g^4 + O(g^6), \quad (16)$$

which is more explicit way of writing (9). For a given E , the ϵ -dependence of g is defined by holding g_R^2 fixed as $\epsilon \rightarrow 0$. The condition

$$\frac{d}{d \log \epsilon} g_R^2 = 0 \quad (17)$$

⁹³Section 21

⁹⁴Montvay and Münster (1997), equation (3.257)

⁹⁵Section 22 will review a renormalization scheme in which the quantity held fixed is defined by starting with a rectangular Wilson loop and taking the Fourier transform with respect to its width in one dimension. Then the variable E is the Fourier conjugate of the width variable.

gives

$$\frac{d}{d \log \epsilon} g^2 = \frac{-f'g^4 + O(g^6)}{1 + 2fg^2 + O(g^4)} = -f'g^4 + O(g^6). \quad (18)$$

For a given ϵ , the quantity g_R is a function of the physical energy scale E . The E -dependence of g_R^2 is given by equation (16):

$$\frac{d}{d \log E} g_R^2 = f'g^4 + O(g^6) = f'g_R^4 + O(g_R^6) \quad (19)$$

If $f' < 0$, then equation (18) says that $g^2 \rightarrow 0$ as $\epsilon \rightarrow 0$, and equation (19) says that $g_R^2 \rightarrow 0$ as $E \rightarrow \infty$. Equations (18) and (19) may differ from each other at higher orders in the small- g expansion,⁹⁶ but at least to lowest order, this shows that the bare and renormalized versions are both controlled by the same function $f(\epsilon E)$: asymptotic freedom occurs if the slope of that function is negative.

⁹⁶Creutz (1983), text below equation (13.17); Smit (2002), equations (5.27)-(5.32)

29 Deducing the slope of $d\beta/d\epsilon$ when $d \neq 4$

What happens to β in the limit $\epsilon/\rho \rightarrow 0$ if a prediction of the form $g_R^2 = g^2 + O(g^4)$ is held fixed in units of the physical length ρ ? Section 12 previewed the answer. This section uses a simple scaling argument to deduce the answer when $d \neq 4$.

The dimensionless expansion parameter is $\beta^{-1} \propto g^2 \epsilon^{4-d}$, so the quantity g_R^2 in units of ρ is⁹⁷

$$g_R^2 \rho^{4-d} = g^2 \rho^{4-d} + O(g^4) \propto \frac{1}{\beta} \left(\frac{\rho}{\epsilon} \right)^{4-d} + O(1/\beta^2)$$

with a proportionality factor independent of β and ϵ . The lowest-order term by itself would say that the function $\beta(\epsilon/\rho)$ defined by holding g_R^2 fixed as $\epsilon/\rho \rightarrow 0$ would be

$$\beta(\epsilon/\rho) \sim (\epsilon/\rho)^{d-4}. \quad (20)$$

The significance of this depends on the number d of spacetime dimensions:

- When $d < 4$, (20) would give $\beta \rightarrow \infty$ as $\epsilon/\rho \rightarrow 0$.⁹⁸ This is the bare version of asymptotic freedom.⁹⁹
- When $d > 4$, the opposite occurs: (20) says that β decreases as $\epsilon/\rho \rightarrow 0$, so the existence of a nontrivial continuum limit would require approaching a nontrivial fixed point, which is beyond the scope of this article.¹⁰⁰
- For $d = 4$, the sign of the slope of the function $\beta(\epsilon/\rho)$ is not determined by the lowest-order term, so the g^4 term must be calculated before a conclusion can be reached. Section 30 will review the result of that calculation.

⁹⁷This section writes the physical length scale as ρ for consistency with most of the preceding sections. We could use $\rho = 1/E$ for consistency with sections 22, 25, and 28. This would only change the notation, not the content.

⁹⁸This is consistent with (but doesn't justify) neglecting the higher-order terms.

⁹⁹When $G = U(1)$, calling this *asymptotic freedom* might be misleading, because the continuum limit of the model with $G = U(1)$ is strictly free (no interactions), not just asymptotically free. This is related to the absence of $O(g)$ terms in equation (4) when $G = U(1)$.

¹⁰⁰Nontrivial fixed points probably don't exist for the Wilson action. Their possible existence for other lattice actions that have the same continuum limit is under investigation (footnote 29 in section 7).

30 Results when $d = 4$

For QCD with N_c colors and N_f flavors in four-dimensional spacetime, the coefficients in equation (11) are¹⁰¹

$$\gamma_4 = \frac{1}{(4\pi)^2} \frac{11N_c - 2N_f}{3} \quad \gamma_6 = \frac{1}{(4\pi)^4} \frac{34N_c^3 - 13N_c^2N_f + 3N_f}{3N_c} \quad (21)$$

Set $N_f = 0$ to get the result for $SU(N_c)$ Yang-Mills theory without quarks:

$$\gamma_4 = \frac{1}{(4\pi)^2} \frac{11N_c}{3} \quad \gamma_6 = \frac{1}{(4\pi)^4} \frac{34N_c^2}{3}. \quad (22)$$

More generally, when the gauged group G is connected and when N_f quark fields are included in a representation r of G , the first coefficient is¹⁰²

$$\gamma_4 = \frac{11}{3}C(\text{adj}) - \frac{4}{3}N_fC(r) \quad (23)$$

where $C(r)$ is defined like ν was in section 13, but using the specified representation r .^{103,104} In the first term, this quantity is defined using the adjoint representation. In the second term, it's defined using the representation r appropriate for the quark fields.¹⁰⁵ The important properties of $C(r)$ are:

- If r is the fundamental representation, then $C(r) > 0$.
- If G is nonabelian, then $C(\text{adj}) > 0$.
- If G is abelian, then $C(\text{adj}) = 0$.

Using these results in equation (11) gives asymptotic freedom when $G = SU(N_c)$.

¹⁰¹Lucini (2013), equation (2); Creutz (1983), equations (12.15), (13.3), and (13.5); Smit (2002), equation (5.38) (for $N_f = 0$)

¹⁰²Peskin and Schroeder (1995), equation (16.134); Elliott *et al* (2018), theorem 5.1

¹⁰³Peskin and Schroeder (1995), equation (15.78)

¹⁰⁴ $C(r)$ is the **quadratic Casimir invariant** for the given representation.

¹⁰⁵This is typically the fundamental representation, so that each quark field has N_c components when $G = SU(N_c)$.

31 The case $G = U(1)$

When the gauged group is $G = U(1)$, a renormalized coupling g_R can still be defined as in section 22, but holding that g_R fixed as $\epsilon/\rho \rightarrow 0$ doesn't give a nontrivial continuum limit.¹⁰⁶ This negative result might be intuitively expected because the continuum action (4) is quadratic (independent of g) when $G = U(1)$.¹⁰⁷ It becomes less intuitive when matter is included, though, as in quantum electrodynamics, because then the continuum action is not quadratic. Still, the available mathematical evidence indicates that the continuum limit of quantum electrodynamics is necessarily trivial (no interactions).¹⁰⁸ That's not a problem for real-world applications, because choosing a value of the interaction strength g_R that is consistent with the real-world electromagnetic interaction is compatible with treating spacetime as a lattice that is much finer than the resolution of any realistic experiment.¹⁰⁹ It's not compatible with taking the lattice spacing all the way to zero in physical units, but we can take the lattice spacing to be effectively zero as far as any realistic experiment can tell.

On a spacetime lattice, the constructions of quantum electrodynamics (QED) and quantum chromodynamics (QCD) are the same except for the choice of gauged group G : QED uses $G = U(1)$, and QCD uses $G = SU(N_c)$. This leads to an overall sign difference in the results that were reviewed in section 30. Section 32 will highlight one consequence of this sign difference. That consequence is a key part of why QCD is believed to have a nontrivial continuum limit and QED is not.

¹⁰⁶What it does give depends on which g_R is used. For the g_R described in section 22, it doesn't give a continuum limit at all: it makes g approach a finite nonzero value (Smit (2002), equations (5.19)-(5.21)), so the correlation length doesn't diverge in units of the lattice spacing, which implies that the correlation length goes to zero in units of ρ when $\epsilon/\rho \rightarrow 0$. We can get a continuum limit by sending $g \rightarrow 0$, but that makes the model trivial (no interactions).

¹⁰⁷Even though this might be intuitively expected, it is not obvious, because the original action (1) is not quadratic.

¹⁰⁸Göckeler *et al* (1998)

¹⁰⁹Quantum electrodynamics doesn't include any of the non-electromagnetic interactions that are known to exist, so it already starts to disagree with experiment at resolutions that are much, much too coarse to be noticeably affected by lattice artifacts (footnote 111 in section 32; McGreevy (2019), section 2.2).

32 QED compared to QCD

Consider the case $d = 4$ (four-dimensional spacetime). Use the abbreviation $x \equiv \log(\epsilon/\rho)$, so $x \rightarrow -\infty$ as $\epsilon/\rho \rightarrow 0$. In QED and QCD, the one-loop approximation to the beta-function is

$$\frac{d}{dx} g^2 = s g^4, \quad (24)$$

with $s > 0$ for QED, and $s < 0$ for QCD with a sufficiently small number of quark flavors.¹¹⁰ The general solution of (24) is $g^2 = 1/(c - sx)$ for some constant c . For a continuum limit, we want $\epsilon/\rho \rightarrow 0$, so suppose we start with $\epsilon/\rho < 1$. The constant c should be such that $g^2 > 0$ for that initial value of ϵ/ρ . What happens as ϵ/ρ decreases depends on the sign of s :

- If $s < 0$, then $c - sx$ is an increasing function of ϵ/ρ . It was positive initially, so it remains positive, and $g^2 \rightarrow 0$ as $\epsilon/\rho \rightarrow 0$.
- If $s > 0$, then $c - sx$ is a decreasing function of ϵ/ρ . It was positive initially, so it crosses through zero at some finite value of x , say x_{LP} , where LP stands for *Landau pole*. Then $g^2 \rightarrow \infty$ ($\beta \rightarrow 0$) as $x \rightarrow x_{\text{LP}}$, which implies that the correlation length for the gauge field goes to zero.

Equation (24) is only approximate (it neglects terms of order g^6 and higher), but it suggests that reaching a nontrivial continuum limit is obstructed when $s > 0$ but not when $s < 0$. The obstruction when $s > 0$ is called a **Landau pole**, and the fact that no such obstruction exists (as far as we know) when $s < 0$ is consistent with the assertion that Yang-Mills theory with a nonabelian gauged group has a nontrivial strict continuum limit.¹¹¹

¹¹⁰Section 30

¹¹¹This can be used to quantify the statement that models like QED can still be very useful for physics despite the absence of a nontrivial strict continuum limit (section 31). Suppose that $g^2 = g_R^2/(1 - s g_R^2 \log(\epsilon/\rho))$, which satisfies equation (24) and is consistent with equation (9). With the values $g_R^2 \sim 0.01$ and $s \sim 0.01$, which are representative for QED (equation (23)), the condition $g^2 \ll 1$ allows $|\log(\epsilon/\rho)|$ to be as large as ~ 100 , which is more than enough to allow ϵ to be much finer than any practically resolvable scale, even though we can't take ϵ/ρ all the way to zero.

33 Different continuum limits when $G = U(1)$

The example in this section illustrates that the properties of a continuum limit can depend on how the limit is taken. This example uses the *Villain model* that was mentioned in section 20.¹¹²

Consider the case $d = 3$ and $G = U(1)$. Let L be the correlation length (related to the mass gap m by $L = 1/m$), and define $\lambda \equiv L/\epsilon$. According to equation (3.1) in Athenodorou and Teper (2019), the dimensionless correlation length approaches¹¹³

$$\lambda^2 = (c\beta)^{-1} \exp(\tilde{c}\beta) \quad (25)$$

as $\beta \rightarrow \infty$, with β -independent constants c and \tilde{c} . When $d = 3$ and $N_{\text{id}} = 1$, equation (3) gives $\epsilon g^2 \propto 1/\beta$, so multiplying (25) by $(\epsilon g^2)^2$ gives

$$(g^2 L)^2 = (\epsilon g^2)^2 \lambda^2 \propto \beta^{-3} \exp(\tilde{c}\beta). \quad (26)$$

Equation (25) shows that the continuum limit $\lambda \rightarrow \infty$ corresponds to $\beta \rightarrow \infty$. Equation (26) shows that $g^2 L \rightarrow \infty$ in that limit, so the limit may be taken in either of two ways:¹¹⁴ we can keep L finite and let $g^2 \rightarrow \infty$, or we can keep g^2 finite (either nonzero or approaching zero) and let $L \rightarrow \infty$. In both cases, the resulting model has a single species of particle, with no interactions.¹¹⁵ The mass of the particle is $\propto 1/L$.^{115,116} The zero-mass case is quantum electrodynamics in three-dimensional spacetime without any matter.¹¹⁷

¹¹²The Villain β and the Wilson β are asymptotically equal for large β (Janke and Kleinert (1986), equation (8)), but those authors also warn that for $d = 4$, the model based on the Villain action is not a good approximation to the one based on the Wilson action near the critical value of β (where a phase transition occurs), which is finite for $d = 4$.

¹¹³Their equation (3.1) is actually for the mass gap (also equation (1.8a) in Göpfert and Mack (1982), equation (5) in Loan *et al* (2003), and equation (2.3) in Caselle *et al* (2015)), which they denote m_D . This is related to the correlation length by $L(\beta) = 1/m_D$ (Göpfert and Mack (1982), text above equation (1.10)).

¹¹⁴No matter how the limit is taken, the quantity ϵg^2 goes to zero because $\beta \propto 1/(\epsilon g^2)$. In other words, g^2 goes to zero when it is expressed in units of $1/\epsilon$.

¹¹⁵Athenodorou and Teper (2019), text below equation (3.3)

¹¹⁶Page 552 in Göpfert and Mack (1982) calls this a conjecture. I don't know why.

¹¹⁷The particle is a photon, which has spin zero when spacetime is three-dimensional (article 26542).

34 Correlation length and the continuum limit

This section derives a relation between lattice spacing and correlation length in units of ϵ that is consistent with the first two terms in the small-coupling expansion¹¹⁸ when $d = 4$ and $G = SU(N_c)$, assuming that the mass gap is nonzero so that the correlation length is finite. The result, equation (32), is similar to the result for $d = 3$ and $G = U(1)$ that was shown in equation (25).

As in that section, let L be the correlation length, and define $\lambda \equiv L/\epsilon$. Equation (11) gives

$$-\frac{1}{2} \frac{dg^2}{d \log \lambda} = \gamma_4 g^4 + \gamma_6 g^6 + O(g^8). \quad (27)$$

To solve this, write it as

$$-\frac{1}{2} \frac{dg^2}{\gamma_4 g^4 + \gamma_6 g^6 + O(g^8)} = d \log \lambda. \quad (28)$$

Using the abbreviation $x \equiv 1/g^2$, this may also be written

$$\frac{1}{2} \frac{dx}{\gamma_4 + \gamma_6/x + O(1/x^2)} = d \log \lambda, \quad (29)$$

which implies

$$\frac{1}{2} \frac{dx}{\gamma_4} \left(1 - \frac{\gamma_6}{\gamma_4 x} + O(1/x^2) \right) = d \log \lambda. \quad (30)$$

Integrating both sides gives

$$\frac{1}{2\gamma_4} \left(x - \frac{\gamma_6}{\gamma_4} \log x + O(1/x) \right) = \log \lambda + \text{const.} \quad (31)$$

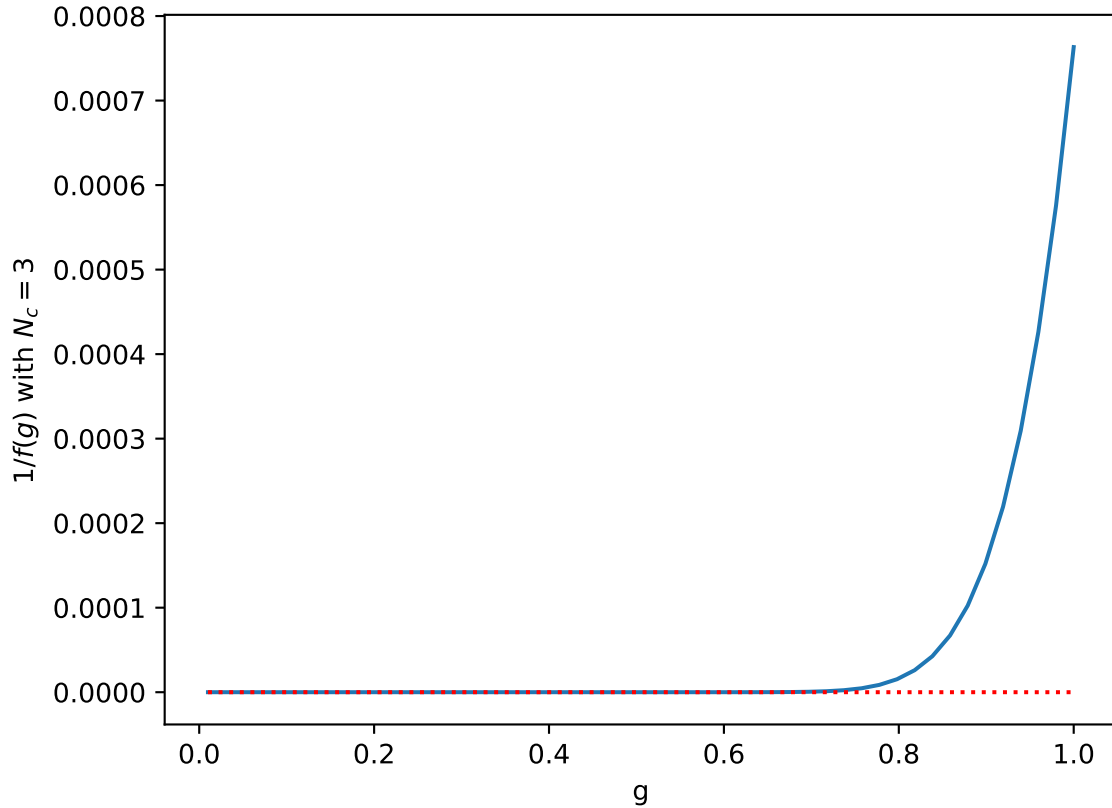
Ignoring the $O(1/x) = O(g^2)$ term, this implies¹¹⁹

$$\lambda \propto f(g) \equiv \exp \left(\frac{1}{2\gamma_4 g^2} \right) g^{\gamma_6/\gamma_4^2}. \quad (32)$$

¹¹⁸This section doesn't address whether the small-coupling expansion is a good approximation.

¹¹⁹Creutz (1983), equation (13.23); Montvay and Münster (1997), equation (3.267); Kogut (1983), equation (4.60); Allés *et al* (1998), equation (14)

Equation (32) says $g \rightarrow 0$ as $\lambda \rightarrow \infty$.¹²⁰ This is consistent with asymptotic freedom. The shape of the function $1/f(g)$ is indicated here, using the values of γ_4 and γ_6 from section 30 with $N_c = 3$ as an example:



The function $f(g)$ diverges very quickly as $g \rightarrow 0$. A few of its values are

$$f(1/2) \approx 1.6 \times 10^{12} \quad f(1/4) \approx 2.4 \times 10^{49} \quad f(1/8) \approx 5.6 \times 10^{198}$$

This shows how rapidly the correlation length grows in units of the lattice spacing as $g \rightarrow 0$, according to equation (32).

¹²⁰ $1/f(g) \rightarrow 0$ as $g \rightarrow \infty$, too, but only the small- g region is considered here because the derivation of (32) assumed that g is small.

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