Phase Structure of Models of Scalar Quantum Fields

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Abstract In quantum field theory, models are often defined with one or more continuously adjustable parameters. Different regions of parameter-space in which the vacuum state has qualitatively different properties are called different **phases**, and the thresholds between those regions are called **phase transitions**. For many of the models that we know how to construct nonperturbatively, the only known nonperturbative constructions involve pretending that spacetime is discrete. Understanding the phase structure of models defined in discrete spacetime can inform the search for interesting continuum limits. This article reviews the phase structure of some simple models of scalar quantum fields, emphasizing how it depends both on the type of internal symmetry and on the number of dimensions of spacetime.

Contents

1 Motive

Roughly, a phase transition is a qualitative change in a system's properties as some parameter is varied. The phase transitions of interest in this article are qualitative changes in the vacuum state induced by varying parameters in the action.

The phase structure of a quantum field model can be important for a few reasons:

- If the model is defined by treating space or spacetime as a lattice,[1] then knowing how to tune the model's parameters so that the correlation length becomes infinite in units of the discretization scale is a prerequisite for the existence of an interesting continuum limit.

- We must be careful that results obtained with some finite lattice spacing are not separated from the continuum limit by some other phase transition across which the results change qualitatively.[2] A famous example of this relates to the question of whether quantum chromodynamics predicts the confinement of quarks. That question has been answered in the limit of strong coupling when spacetime is treated as a lattice, but this doesn't prove that quarks remain confined when the continuum limit is taken, because a phase transition might occur along the way.[3,4]

- It can be used to test intuition. Intuition can be valuable, but it isn't always reliable, so it should be tested in a variety of ways. Information about a model's phase structure – and how the phase structure depends on basic inputs like the number of dimensions of spacetime – can provide some of those tests.

[1] Articles 52890 and 63548

[2] Svetitsky (2013) says, "No lattice calculation... can proceed without understanding the phase diagram..."

[3] Creutz (1980), section I

[4] Numerical computations overwhelmingly indicate that such phase transitions are absent and that quarks really are confined in the continuum limit of QCD, as they are in nature. The *asymptotic freedom* phenomenon (section 13) is also consistent with this.

2 Sources of information

This article summarizes a variety of published results from a variety of sources. Many of these results were derived using approximations,[5] but all of them have passed enough consistency checks to be convincing. Aside from that generic disclaimer, the most important thing to understand about these results is that many of them have been *transferred*. The next paragraph explains what I mean by this.

The euclidean path integral formulation for vacuum expectation values of a model of scalar quantum fields in d-dimensional spacetime has the same mathematical form[6] as statistical expectation values over a classical Boltzmann distribution in d-dimensional space. The phase structure is the same either way, with the understanding that vacuum (zero-temperature) expectation values in the quantum case correspond to thermal expectation values in the classical case. Using this correspondence, results about the phase structure of classical models can often be *transferred* to the corresponding quantum models. The language and notation may differ, but the math is the same. The language in this article is focused on quantum models, even though many of the cited sources are written with the classical case in mind. The results are transferrable.[7]

[5]The details of the approximations can be found by studying the cited sources.

[6]Article 63548

[7]The quantum-classical correspondence reviewed in this paragraph is not the only type of quantum-classical correspondence. Article 51033 includes a long section about this. Of course, we *cannot* necessarily transfer results based on those other types of quantum-classical correspondence. Remember that in the type of correspondence used in this article, the number of *spacetime* dimensions in the quantum model is equal to the number of *space* dimensions in the corresponding classical model.

3 The continuum and infinite-volume limits

The quantum models in this article may be constructed using the path integral formulation (section 4) by treating spacetime as a lattice. One of the motivations for studying the phase structure of these models is to inform the search for interesting continuous-spacetime limits, but the quoted results are not restricted to those limits.

In contrast, the limit of *infinite volume* (infinite number of lattice sites) is understood to be in effect throughout this article. That's important because some types of phases and phase transitions occur only in infinite volume, at least as far as the model itself can tell.[8]

[8]Real systems can exhibit phenomena like spontaneous symmetry breaking in finite volume (article 81040), for the same reason that real systems can exhibit the phenomena we call *measurement* (article 03431). Another perspective is described in van de Ven *et al* (2020).

4 Wick rotation and phase structure

Consider a model involving a scalar field in d-dimensional spacetime, which will be treated as a lattice, and let I be some time-ordered product of field operators and their expectation values.[9] This article focuses on models in which the vacuum expectation value of the operator I can be reconstructed from the **euclidean path integral**[10]

$$\langle I\rangle \equiv \frac{\int [d\phi]\ e^{-S[\phi]} I[\phi]}{\int [d\phi]\ e^{-S[\phi]}} \tag{1}$$

using Wick rotation.[11] The insertion $I[\phi]$ (which represents the operator I) and the euclidean action $S[\phi]$ are both expressed in terms of the scalar field variables $\phi(x)$ and their discretized derivatives with respect to x. In this article the euclidean action $S[\phi]$ will just be called the **action**.

[9]Mnemonic: I stands for *insertion* or *integrand*, because of the way this operator is represented in the path integral (1).

[10]Article 63548

[11]If the result of evaluating the path integral is expressed as a function of the time-step dt, then the vacuum expectation value of I is recovered by replacing every occurrence of dt with $i\,dt$ (if the fields are all scalar fields). The name **Wick rotation** refers to writing this as $e^{i\theta}dt$ and then "rotating" θ continuously from 0 to $\pi/2$.

5 The ϕ^4 model: phase structure

The euclidean action of the ϕ^4 model is[12,13]

$$S = \epsilon^d \sum_{\{x,y\}} \frac{1}{2}\left(\frac{\phi(y)-\phi(x)}{\epsilon}\right)^2 + \epsilon^d \sum_x \left(\mu\frac{\phi^2(x)}{2} + g\frac{\phi^4(x)}{4!}\right) + \text{constant}, \tag{2}$$

where the sum over $\{x, y\}$ means the sum over nearest-neighbor pairs of points in spacetime, which is treated as a lattice. The action (2) and the measure $[d\phi]$ in the path integral are both invariant under the transformation $\phi(x) \to -\phi(x)$, so this is a symmetry.[14] The ϕ^4 model with $g > 0$ can be in either of two phases, depending on the values of the coefficients in (2) and the value of d:

- In the **symmetric phase**, the vacuum state is unique.
- In the **SSB phase**,[15] two choices exist for the vacuum state, related to each other by the discrete symmetry $\phi \to -\phi$. In other words, the $\phi \to -\phi$ symmetry is spontaneously broken. This phase exists only if $d \geq 2$.

At the phase transition, the correlation length becomes infinite in units of the lattice spacing,[16] satisfying a necessary (but not sufficient) condition for the existence of an interesting continuum limit. For any given value of the coefficient g in (2), the value of μ that makes the correlation length infinite is called the **critical point**, denoted μ_c. The function $\mu_c(g)$ defines a line in the g-μ plane called the **critical line**. Section 6 uses some intuition to deduce that μ_c becomes increasingly negative as g increases, and sections 7 and 9 show some results from computer calculations that confirm this intuition.

[12]Article 63548

[13]For the rest of this article, constants terms in the action will be omitted even if their values change from one equation to the next.

[14]It's an **internal symmetry**, because it doesn't mix the field variables at different points with each other.

[15]*SSB* stands for *spontaneous symmetry beaking*.

[16]in many models with an SSB phase, the correlation length becomes infinite at the phase transition, but exceptions do exist. In the *q-state Potts model* with $q \geq 5$ and $d = 2$, the correlation length remains finite at the transition to the SSB phase (Creswick and Kim (1997)).

6 The ϕ^4 model: correlation length

Consider the ϕ^4 model with $g = 0$ and $\mu = m^2 > 0$. Then the correlation length is inversely proportional to the parameter m. To understand intuitively why this is true, start with the euclidean action:

$$S = \epsilon^d \sum_{\{x,y\}} \frac{1}{2} \left(\frac{\phi(y) - \phi(x)}{\epsilon} \right)^2 + \epsilon^d \sum_x m^2 \frac{\phi^2(x)}{2}. \tag{3}$$

Rescale $\phi \to \phi/\sqrt{m}$ to get

$$S = \epsilon^d \sum_{\{x,y\}} \frac{1}{2} \left(\frac{\phi(y) - \phi(x)}{m\,\epsilon} \right)^2 + \epsilon^d \sum_x \frac{\phi^2(x)}{2}. \tag{4}$$

This shows that correlations between different lattice sites are suppressed as $m \to \infty$, so large m means small correlation length. This agrees with the result derived in article 00980, but the intuition used here also works in cases like this:

$$S = \epsilon^d \sum_{\{x,y\}} \frac{1}{2} \left(\frac{\phi(y) - \phi(x)}{\epsilon} \right)^2 + \epsilon^d \sum_x +g \frac{\phi^4(x)}{4!}. \tag{5}$$

The same kind of rescaling argument, now with $\phi \to \phi/g^{1/4}$, says that larger g gives smaller correlation length. Finally, consider

$$S = \epsilon^d \sum_{\{x,y\}} \frac{1}{2} \left(\frac{\phi(y) - \phi(x)}{\epsilon} \right)^2 + \epsilon^d \sum_x \left(\mu \frac{\phi^2(x)}{2} + g \frac{\phi^4(x)}{4!} \right). \tag{6}$$

Now we have two parameters, μ and g. Intuitively, if we make the correlation length smaller by increasing g, then we should be able to make it larger again by decreasing μ. If μ is initially zero, then keeping the correlation length large must require making μ more *negative* as g increases. Numerical studies show that this intuition is correct. Some results are summarized in sections 7 and 9.

7 The ϕ^4 model: critical line

If $d \geq 2$, then for any given value of $g > 0$, a finite critical value $\mu_c < 0$ exists for which the model with action (2) has a phase transition at $\mu = \mu_c$. The case $\mu < \mu_c$ gives the SSB phase, and $\mu > \mu_c$ gives the symmetric phase. The correlation length is finite for all μ except $\mu = \mu_c < 0$, where it becomes infinite. This table shows the critical value μ_c for a few representative values of d and g:[17]

d	g	μ_c
3	0	0
	46	-9.2
	∞	$-\infty$
4	0	0
	150	-8
	∞	$-\infty$

This is consistent with the intuition in section 6.

When $g = 0$, the would-be SSB phase doesn't exist, because the path integral is undefined when $\mu < 0$ unless $g > 0$.[18]

Section 8 introduces a different way of writing the action, one that clarifies what happens when $g \to \infty$.

[17]The computer results are shown to two significant digits. This table is inferred from the one in section 9 using equations (7). Article 10142 shows a more complete table for the case $d = 4$.

[18]When $\mu < 0$ and $g = 0$, the factor e^{-S} in the integrand of the path integral does not have a finite upper bound.

8 Another way to write the action

To reduce clutter, use units in which $\epsilon = 1$ from now on. Define λ and κ by

$$g = \frac{6\lambda}{\kappa^2} \qquad\qquad \mu = \frac{1-2\lambda}{\kappa} - 2d, \tag{7}$$

and rescale the field $\phi \to \sqrt{2\kappa}\,\phi$. Then the action (2) may be written as

$$S = -2\kappa \sum_{\{x,y\}} \phi(x)\phi(y) + \sum_x \left(\phi^2(x) + \lambda\Big(\phi^2(x) - 1\Big)^2\right). \tag{8}$$

The coefficient κ is called the **hopping parameter**, because the product $\phi(x)\phi(y)$ is what allows effects to propagate ("hop") through the lattice.

The original action (2) has two adjustable parameters, μ and g. The new action (8) still has two adjustable parameters, κ and λ, but now a simplification occurs in the limit $\lambda \to \infty$: in that limit, the factor e^{-S} in the integrand of the euclidean path integral goes to zero unless $\phi(x) = \pm 1$ for all x, so the action can be reduced to

$$S_{\lambda\to\infty} = -2\kappa \sum_{\{x,y\}} \phi(x)\phi(y) \qquad\qquad \text{with } \phi(x) = \pm 1.$$

This is called the **Ising model**.

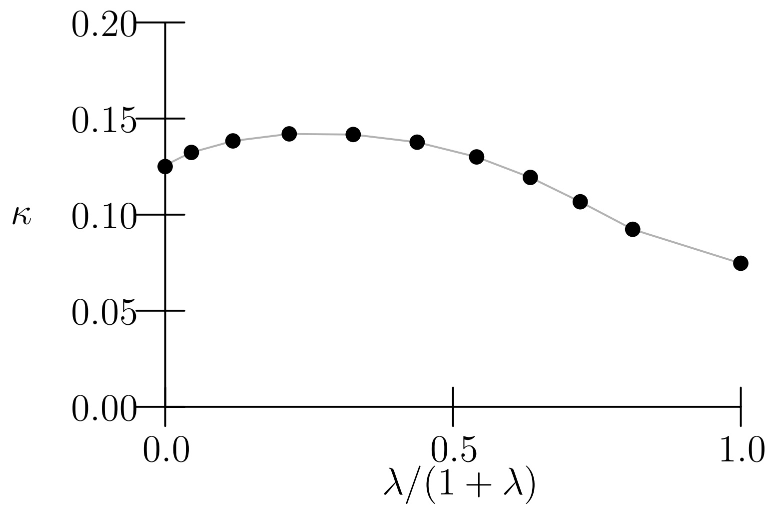
When the model's phase structure is described using the parameters in the original action (2), the critical value of μ goes to $-\infty$ as $g \to \infty$. Describing the model's phase structure in terms of κ and λ gives a more complete picture, because the value of κ at which the phase transition occurs – the critical value κ_c – remains finite as $\lambda \to \infty$. Section 9 will describe the critical line again, using κ and λ instead of μ and g.

9 The critical line again

This table shows the critical value of 2κ for models with the action 8, for a few representative values of d and λ:

d	λ	$2\kappa_c$	Sources
3	1.1	0.38	Hasenbusch (1999)
	∞	0.22	Creutz (1983)
4	0.49	0.28	Lüscher and Weisz (1987)
	∞	0.15	Lüscher and Weisz (1987), Lüscher (1987)
$\to\infty$	∞	$\tanh^{-1}\frac{1}{2d-1}$	Itzykson and Drouffe (1989)

As $\lambda \to 0$, the critical value approaches $\kappa_c \to 1/d$, but the SSB phase doesn't exist when $\lambda = 0$ because the model is undefined for $\kappa > 1/d$ when $\lambda = 0$.[19] The value of κ_c as a function of $\lambda/(1+\lambda)$ is shown here for the case $d = 4$:[20]



For values of κ below the line, the model is in the symmetric phase. For values of κ above the line, it's in the SSB phase (for $\lambda > 0$).

[19] Footnote 18 in section 7

[20] The dots are data-points from table 1 in Lüscher and Weisz (1987). The gray line between the dots is naïve linear interpolation. The kink at the second-to-last dot is an artifact of this naïve interpolation.

10 The $O(N)$ models

The family of models described in the previous sections has a natural generalization in which the scalar field has N components:

$$\phi(x) = \big(\phi_1(x), ..., \phi_N(x)\big).$$

As in section 8, the euclidean action is

$$S = -2\kappa \sum_{\{x,y\}} \phi(x)\cdot\phi(y) + \sum_x \left(\phi^2(x) + \lambda\Big(\phi^2(x) - 1\Big)^2\right) \tag{9}$$

with

$$\phi(x)\cdot\phi(y) \equiv \sum_{n=1}^{N} \phi_n(x)\phi_n(y) \qquad\qquad \phi^2(x) \equiv \phi(x)\cdot\phi(x).$$

The action is invariant under x-independent rotations in the N-dimensional space whose coordinates are the N components of the scalar field. These are called $\boldsymbol{O(N)}$ **models**, named after this $O(N)$ symmetry. When $N = 1$, this reduces to action (8) for the ϕ^4 model.

In the limit $\lambda \to \infty$, the factor e^{-S} in the integrand of the euclidean path integral goes to zero unless the field satisfies the constraint $\phi^2(x) = 1$ for all x, so the action in the path integral can be reduced to[21]

$$S = -2\kappa \sum_{\{x,y\}} \phi(x)\cdot\phi(y) \qquad\qquad \text{with } \phi^2(x) = 1. \tag{10}$$

This is an example of a **nonlinear sigma model**.[22] The next few sections will review the phase structure of these models.

[21] Article 51033 constructed these models directly, instead of starting with finite λ.

[22] The word *nonlinear* is in the name because of the nonlinear constraint $\phi^2(x) = 1$. Models with unconstrained scalar field variables, like (9), are sometimes called **linear sigma models**, even though the action is a nonlinear function of the field variables.

11 The $O(N)$ models: phase structure

The family of models with action (10) has one continuously adjustable parameter κ, and it has two discrete parameters: the number d of spacetime dimensions, and the number N of components of the scalar field. This table shows how the number of distinct phases depends on d and N[23] when $\kappa > 0$:[24]

	$d = 1$	$d = 2$	$d \geq 3$
$N = 1$	1	2	2
$N = 2$	1	2	2
$N \geq 3$	1	1	2

In cases with more than one phase, the value of κ controls which phase the model is in.

When $d \geq 3$, the model always has two phases: for each N, a critical value $\kappa_c > 0$ exists for which the model is in the symmetric phase for $\kappa < \kappa_c$ and in the SSB phase for $\kappa > \kappa_c$. When $N \geq 2$, the $O(N)$ symmetry is continuous, and a continuum of choices exists for the vacuum state in the SSB phase. This leads to massless particles called **Goldstone bosons**,[25] a phenomenon that does not occur when the symmetry is discrete ($N = 1$).

For $d = 2$, a compact continuous group of symmetries, like $O(N)$ with $N \geq 2$, cannot be spontaneously broken.[26] Section 13 will summarize the nature of the phases in these cases.

[23]For $N \geq 3$ with $d = 2$: Zinn-Justin (1998). For $N \geq 3$ with $d \geq 3$: Roomany and Wyld (1980) and page 678 in Zinn-Justin (1996). For $N = 2$: section 13. For $N = 1$: section 5.

[24]For any d, we can use a spacetime lattice with this property: its points can be assigned to two subsets such that every nearest-neighbor pair has exactly one point from each subset. In that case, only $\kappa \geq 0$ needs to be considered, because changing the sign of κ is equivalent to changing the sign of $\phi(x)$ for points x in one of those two subsets. If a term proportional $\sum_x \phi(x)$ is added to the action, then $\kappa > 0$ and $\kappa < 0$ are no longer equivalent, and new phases can occur when $\kappa < 0$. Section 18 describes one example.

[25]Derivations are given in chapter 19 in Weinberg (1996), page 388 in Peskin and Schroeder (1995), and section 5.3 in Cheng and Li (1984). Chapter 19 in Weinberg (1996) discusses some special properties of Goldstone boson interactions.

[26]This is the **Mermin-Wagner theorem** (article 37301).

12 The $O(N)$ models: critical point

For models with the $O(N)$-symmetric action (10), this table shows approximate values of $2\kappa_c$ (the value of 2κ at which the phase transition occurs) in some representative cases:[27]

	$d=1$	$d=2$	$d=3$	$d=4$	$d\to\infty$
$N=1$	∞	0.44	0.22	0.15	$\tanh^{-1}\frac{1}{2d-1}$
$N=2$	∞	1.1	0.45	0.3	
$N=3$	∞	∞	0.69		
$N=4$	∞	∞	0.94	0.61	
$N=5$	∞	∞			
$N=6$	∞	∞	1.4		
$N=7$	∞	∞			
$N=8$	∞	∞	1.9		
$N=9$	∞	∞			
$N=10$	∞	∞	2.4		
$N=11$	∞	∞			
$N=12$	∞	∞	2.9		
$N\to\infty$		∞	finite$\times N$		$N/2d$

Notice the trend: when d decreases or when N increases, spontaneous symmetry breaking becomes "more difficult," in the sense that it requires a larger value of κ. When d is too small (for a given N) or when N is too large (for a given d), SSB becomes so difficult that it cannot occur at all ($\kappa_c=\infty$).

[27] For $N=1$: page 95 in Creutz (1983) and page 123 in Itzykson and Drouffe (1989). For $d=3$: Butera and Comi (1997). For $N=\infty$ (the **Stanley model**): see pages 14,15, and 143 in Itzykson and Drouffe (1989). For $d=N=2$: page 8 in Butera and Comi (1996) or section 3 in Caselle *et al* (2019). (The latter source gives the value of $1/\kappa_c$.) For $d=4$ with $N=2$: Bock *et al* (1999). For $d=N=4$: Nishimura (1992).

13 The phase structure for $d = 2$

When $d = 2$, the phase structure of a model with action (10) depends on the value of N:

- If $N = 1$,[28] the model has two phases: a symmetric phase for small κ, and an SSB phase for large κ, separated by a finite critical value $\kappa_c > 0$, just like when $d \geq 3$.

- If $N = 2$, the model has two distinct phases, even though it doesn't have an SSB phase. The phase transition in this case is called the **Berezinskii-Kosterlitz-Thouless (BKT) transition**.[29] The correlation length is finite in the small-κ phase and is infinite in the large-κ phase.[30]

- If $N \geq 3$, the critical value of κ is infinite: the correlation length diverges (in units of the lattice spacing) as $\kappa \to \infty$. The model has only one phase, and it has a property called **asymptotic freedom**,[31] which means (roughly) that the interaction strength effectively approaches zero at asymptotically high energies even though it nonzero at finite energies. Asymptotic freedom is interesting partly because it also occurs in quantum chromodynamics when $d = 4$, a discovery that led to a Nobel Prize[32] because it explains why quarks behave almost like free particles at short enough distances even though they interact strongly with each other at larger distances.

[28] Article 81040 derives the phase structure of this model in detail, using a hamiltonian formulation.

[29] It has also often been called the **Kosterlitz-Thouless (KT) transition**. Chapter 4 in Le Bellac (1991) gives some intuition about the nature and reason for this phase transition.

[30] Caselle *et al* (2019), section 3

[31] Section 13.3 in Peskin and Schroeder (1995) reviews two different derivations of this. Patrascioiu and Seiler (1999) encourage thinking critically about the evidence on which this conclusion is based – a healthy exercise even if it doesn't end up changing the conclusion.

[32] https://www.nobelprize.org/prizes/physics/2004/summary/

14 Universality near the phase transition

Some details of a model's construction that affect the model's predictions when spacetime is treated as a lattice can stop affecting its predictions in the continuum limit. This is called **universality**. Taking a continuum limit requires tuning a model's parameters so that its correlation length becomes infinite in units of the lattice spacing (though it may still be kept finite in physically relevant units), so universality can also be expressed like this: when the correlation length diverges (in units of the lattice spacing), as it often does at transitions between different phases, certain quantities no longer depend on the model's microscopic details.[33] Examples of such universal quantities include the **critical exponents** β, ν, and η defined by[34]

$$\text{(vacuum expectation value of the field)} \sim |\kappa - \kappa_c|^{\beta} \tag{11}$$

$$\text{(correlation length)} \sim |\kappa - \kappa_c|^{-\nu} \tag{12}$$

$$\text{(two-point correlation function at } \kappa = \kappa_c\text{)} \sim \frac{1}{(\text{distance})^{d-2+\eta}}. \tag{13}$$

The values of these critical exponents in a few models are listed in this table:[35]

d	N	β	ν	η	Sources
2	1	1/8	1	1/4	p 398 in Huang (1987)
3	1	0.32	0.63	0.03	p 318 in Itzykson and Drouffe (1989), p 636 in Zinn-Justin (1996), p 398 in Huang (1987)
	2	0.35	0.67	0.03	p 318 in Itzykson and Drouffe (1989), p 636 in Zinn-Justin (1996)
	3	0.36	0.71	0.03	p 318 in Itzykson and Drouffe (1989), p 636 in Zinn-Justin (1996), p 398 in Huang (1987)
	∞	1/2	1	0	pp 130, 281 in Itzykson and Drouffe (1989)
4	any	1/2	1/2	0	p 236 in Amit (1997)
∞	any	1/2	1/2	0	p 130 in Itzykson and Drouffe (1989)

[33]In contrast, the value of κ_c is only meaningful in the context of a specific lattice model. This article includes tables of κ_c because the way it depends on discrete parameters like d and N can be used to check intuition.

[34]Huang (1987)

[35]The values for $d = 3$ with finite N are approximate. The others are exact.

15 Z_n models

Article 51033 constructed a family of models called $\boldsymbol{Z_n}$ **models**, one for each integer $n \geq 2$. The $O(2)$ model can be viewed as a limit of the Z_n model as $n \to \infty$. Conversely, the Z_n model can be viewed as a discretized version of the $O(2)$ model in which the allowed values of the field variables correspond to n equally-spaced points around the unit circle, like the tick-marks on a clock. For this reasons, Z_n models are also called **clock models**.

For $d \geq 3$, the phase structure is qualitatively the same for $n \to \infty$ as it is for $n = 2$: the $O(2)$ and Z_2 models both have two phases, a symmetric phase and an SSB phase,[36] so we would naturally expect this same phase structure for every n.

In contrast, for $d = 2$, the Z_2 model still has both a symmetric phase and an SSB phase, but the $O(2)$ model has two distinct symmetric phases instead.[37] This raises a question: when $d = 2$, as n increases, how does the phase structure of the Z_n model turn into that of the $O(2) = Z_\infty$ model? The answer is interesting:[38]

- for $5 \leq n < \infty$, the Z_n model has *three* phases: a symmetric phase for $\kappa < \kappa_c^{(1)}$, a phase with infinite correlation length for intermediate values $\kappa_c^{(1)} < \kappa < \kappa_c^{(2)}$, and an SSB phase for large $\kappa > \kappa_c^{(2)}$,

- For $2 \leq n \leq 4$, the Z_n model has only two phases, symmetric and SSB, because $\kappa_c^{(1)} = \kappa_c^{(2)}$.

- As n approaches ∞, the value of $\kappa_c^{(2)}$ diverges, so the SSB phase disappears in the limit $n \to \infty$ (the $O(2)$ model). The remaining phases are both symmetric, one with finite correlation length and one with infinite correlation length, as described in section 13.

Section 16 shows the approximate values of $\kappa_c^{(1)}$ and $\kappa_c^{(2)}$.

[36] Section 11

[37] Section 13

[38] Ortiz *et al* (2012), section 4, figure 4; Alfonso (1985), section III.2; and Hostetler *et al* (2021), section IV-E, which includes a long list of references.

16 Z_n models: critical points(s) for $d = 2$

The action for the Z_n model may be written

$$S = -2\kappa \sum_{\{x,y\}} \cos\big(\theta(x) - \theta(y)\big),$$

where $\theta(x)$ is restricted to integer multiples of $2\pi/n$. This table shows approximate critical points for the quantum Z_n models in two-dimensional spacetime (or classical Z_n models in two-dimensional space) with this action, in terms of $\beta \equiv 2\kappa$:[39,40]

n	$\beta_c^{(1)}$	$\beta_c^{(2)}$
2	0.44	$= \beta_c^{(1)}$
3	0.67	$= \beta_c^{(1)}$
4	0.88	$= \beta_c^{(1)}$
5	1.05	1.10
6	1.11	1.43
7	1.11	1.88
8	1.12	2.35
⋮		
12	1.12	5.06
⋮		
17	1.11	10.13
⋮		
$\to \infty$	1.12	$\propto n^2$

The significance of the $\beta_c^{(2)}$ column was highlighted in section 15.

[39]For $n = 2$: footnote 27 in section 12, because this case is the same as the $O(N)$ model with $N = 1$. For $n = 2, 3, 4$: Li *et al* (2021), figure S5 (and table S1 cites more references for $n = 5, 6, 7, 8$). For $n = 5$: Borisenko *et al* (2011a), between equations (3.4) and (3.5). For $n = 6, 8, 12$: Tomita and Okabe (2002), table 1 (with $2\kappa = 1/T$). For $n = 7, 17$: Borisenko *et al* (2011b), section 2. For $n \to \infty$: Borisenko *et al* (2011b), section 3.

[40]The notation β is often used when thinking of the euclidean path integral as the statistical sum over a classical Boltzmann distribution (section 2), in which case β is the inverse temperature.

17 The icosahedron model

In the $O(3)$ model, the values of the field variables are restricted to the sphere S^2 defined by $\phi_1^2(\mathbf{x}) + \phi_2^2(\mathbf{x}) + \phi_3^2(\mathbf{x}) = 1$. The **icosahedron model** is like the $O(3)$ model, but the values of the field variables are further restricted to the 12 vertices of a regular icosahedron, a discrete subset of S^2. The resulting symmetry group is the largest subgroup of $O(3)$ that is not also a subgroup of $O(2)$.[41]

When $d = 3$, the icosahedron model has two phases separated by a critical value κ_c.[42] This phase structure is similar to that of the $O(3)$ model when $d = 3$.[43] Hasenbusch (2020) presents evidence[44] that the icosahedron and $O(3)$ models are in the same universality class[45] (they have the same continuous-spacetime limits) when $d = 3$, which implies that the symmetry of the icosahedron model is enhanced to $O(3)$ symmetry in the continuous-spacetime limit.[46]

When $d = 2$, the icosahedron model again has two phases,[47] even though the $O(3)$ model has only one phase, namely the asymptotically-free phase.[43,48]

[41] Caracciolo *et al* (2001), appendix A

[42] Section IV in Hasenbusch (2020) gives the value of κ_c.

[43] Section 11

[44] The numerical analyses reported in Hasenbusch (2020) collectively used more than 130 years of CPU time.

[45] Article 10142 introduces the concept of **universality**.

[46] The continuous-spacetime limit involves only arbitrarily low resolution compared to the lattice spacing, which implies an averaging effect. Averages of sufficiently large numbers of discrete variables can act effectively like continuous variables, so the reported symmetry enhancement at the critical point doesn't contradict the fact that the field variables at each lattice site are restricted to a discrete subset.

[47] Hasenbusch (2020), section II-B

[48] ...at least this is the consensus among most physicists, based on the available evidence. Numerical results using spacetime lattices of currently-attainable sizes seem to indicate that the continuum limit of the icosahedron model, when approached from the symmetric phase, is the same as the $O(3)$ model even when $d = 2$. Patrascioiu and Seiler (2000) suggested that the $d = 2$ $O(3)$ model might not be asymptotically free (contrary to conventional wisdom), and Hasenfratz and Niedermayer (2001a,b) suggested the opposite – that the $d = 2$ icosohedron model might have a continuum limit with asymptotic freedom. Based on a perturbative analysis, Caracciolo *et al* (2001) suggested that the two models may not share the same continuum limt, even though detecting the difference may require considering much larger spacetime lattices.

18 The Z_2 model with explicit symmetry breaking

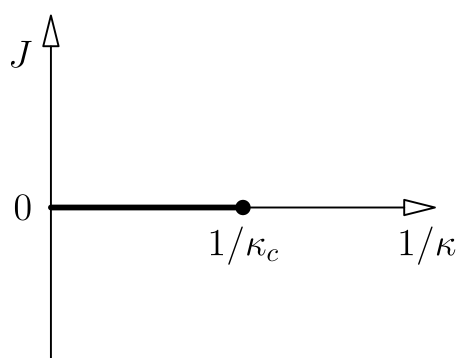
The Z_2 model is named after its symmetry $\phi \to -\phi$, which is broken spontaneously when $\kappa > \kappa_c$. The previous sections considered only nonnegative values of κ, because negative values don't give anything new: changing the sign of κ is equivalent to changing the sign of exactly one of the variables $\phi(x)$ in every nearest-neighbor pair $\{x, y\}$.[49]

This section considers the effect of including an explicit symmetry-breaking term, so that the action is

$$S = -2\kappa \sum_{x,y} \phi(x)\phi(y) - J \sum_{x} \phi(x) \qquad \text{with } \phi(x) = \pm 1. \tag{14}$$

When $J \neq 0$, the transformation $\phi \to -\phi$ is no longer a symmetry of the action, and the phase structure with $\kappa < 0$ is different than the phase structure with $\kappa > 0$.

The phase diagram when $\kappa > 0$ is sketched here:[50]



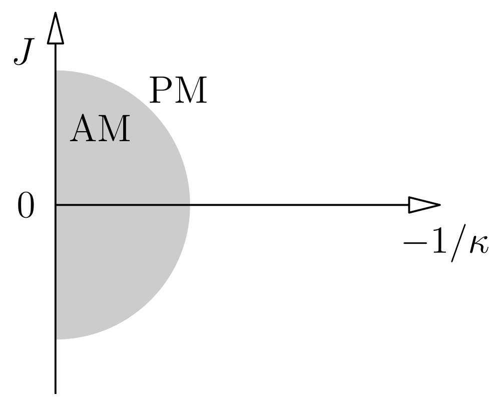
Let v denote the vacuum expectation value of the operator corresponding to $\phi(x)$. When $J \neq 0$, v is nonzero and has the same sign as J. For a fixed value of

[49]This is possible when a particular type of spacetime lattice is used (footnote 24 in section 11), namely the d-dimensional version of a square/cubic/hypercubic lattice.

[50]This is figure 2.4b in Landau and Binder (2015). In that source, the horizontal axis is labelled T, because that source interprets the quantity (14) as the hamiltonian of the classical Ising model in 2-dimensional space (instead of the euclidean action of the quantum Ising model in 2-dimensional spacetime), in which case $T \propto 1/\kappa$ is the temperature.

$1/\kappa < 1/\kappa_c$, v approaches a nonzero limit as $J \to 0$, so it jumps discontinuously (because its sign changes) when the thick line is crossed. We usually call this a phase transition, even though each phase can be reached from the other, without any discontinuous change in properties, by passing *around* the thick line instead of across it.[51] This is possible because the thick line, across which discontinuous changes occur, terminates at a critical point at a finite value of $1/\kappa$.

The case $\kappa < 0$ is sometimes called the **antiferromagnetic** Ising model. When $J = 0$, the factor e^{-S} in the integrand of the path integral favors configurations in which neighboring variables $\phi(x)$ and $\phi(y)$ have the same sign if $\kappa > 0$, or opposite signs if $\kappa < 0$. The name *antiferromagnetic* for the $\kappa < 0$ case refers to the fact that it favors configurations in which nearest neighbors have opposite signs. The phase diagram when $\kappa < 0$ is sketched here:[52]



Two different phases are indicated: the antiferromagnetic (AM) phase, and the paramagnetic (PM) phase. When $J = 0$, replacing $\kappa \to -\kappa$ exchanges the PM ($\kappa < 0$) and symmetric ($\kappa > 0$) phases, and it exchanges the AM ($\kappa < 0$) and SSB ($\kappa > 0$) phases.[53] The value of κ at which the PM-AM transition occurs is the negative of the value of κ at which the symmetric-SSB transition occurs.

[51] This is similar to the relationship between the liquid and vapor phases of water: each one can be reached from the other without any discontinuous change in properties (without boiling or condensing) by following a suitable path in the pressure-temperature plane (article 73054).

[52] Landau and Binder (2015), figure 2.4c

[53] The paragraph after equation (2.18) in Bock *et al* (1999) highlights this same pattern in the $O(N)$ model with $N = 2$ and $d = 4$.

19 Phase structure of principal chiral models

Nonlinear sigma models like the ones considered in section 10 are sometimes called **chiral**[54] models because of their use as low-energy effective models of Goldstone bosons associated with spontaneously broken chiral symmetries.[55] Article 51033 constructed a special family of nonlinear sigma models called **principal chiral models**, in which the target space (the space of allowed values of the field variables) is a Lie group G.

According to Cherman *et al* (2014), the principal chiral model with with $d = 2$ and $G = SU(N)$ is asymptotically free,[56] like the models in section 13 with $N \geq 3$.

The $O(2)$ model, whose phase structure was reviewed in sections 11 as a function of d, is a principal chiral model with $G = SO(2)$.[57]

I have not found any good information about the phase structure of the simplest principal chiral models when spacetime has $d \geq 3$ dimensions and G is nonabelian.[58] What I have seen in the literature is a bogus argument that these models must be in the SSB phase.[59] The key step in that bogus argument goes like this: the fact that the values of the field variables are constrained to be nonzero implies that the vacuum expectation value of the corresponding operators must also be nonzero. Maybe authors who use this argument have additional conditions in mind that would make the conclusion conditionally valid, but without those conditions, the argument is bogus. This article reviewed several counterexamples.[60] Simple intuition can be valuable, but beware of intuition that doesn't pass simple checks.

[54] Example: Zamolodchikov and Zamolodchikov (1979)

[55] Tong (2018), section 5.2

[56] Kazakov *et al* (2020) cites more references about principal chiral models with $d = 2$ and $G \in \{SU(N), SO(N)\}$.

[57] For most N, the $O(N)$ models with target space S^{N-1} are not principal chiral models, because S^{N-1} is not homeomorphic to any Lie group unless $N \in \{1, 2, 4\}$ (`https://math.stackexchange.com/questions/12453/`). The Lie group homeomorphic to S^3 is $SU(2)$, which can also be described as the multiplicative group of **versors** (unit quaternions).

[58] Maybe good information exists that I haven't found yet. Principal chiral models are not my specialty.

[59] Such models may indeed have an SSB phase, at least when $d \geq 3$, but the bogus reasoning would suggest that this is their *only* phase (even when $d \leq 2$), which is incorrect.

[60] One counterexample is the Z_2 model (section 15): for any $d \geq 2$, the model has a symmetric phase in which the vacuum expectation value of the field operator is zero, even though the field variables are constrained to be nonzero.

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