Clifford Algebra, also called Geometric Algebra

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Abstract This article introduces **Clifford algebra**, an associative algebra generated by vectors. Like the exterior algebra (article 81674), it includes objects representing k-dimensional subspaces for every k. Unlike the exterior algebra, though, Clifford algebra includes a scalar product of two vectors, like the one defined by the metric tensor at a given point in spacetime. For this reason, Clifford algebra is sometimes also called **geometric algebra**.

Contents

1 Tensor algebra

Let $\mathcal{V}$ be a d-dimensional[1] vector space[2] over a given field of **scalars**, which may be either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. For the rest of this article, individual vectors (elements of $\mathcal{V}$) will be denoted by boldface lowercase letters like $\mathbf{a}$ and $\mathbf{b}$.

The definition of Clifford algebra starts with the definition of **tensor algebra**. Informally: the tensor algebra of $\mathcal{V}$, denoted $T(\mathcal{V})$, is the largest[3] associative algebra[4] generated by the vectors in $\mathcal{V}$ and a multiplicative unit element 1, over the given field of scalars.

Here's a more tangible (but still informal) way to define $T(\mathcal{V})$. Given a set of d linearly independent elements of $\mathcal{V}$, the tensor algebra $T(\mathcal{V})$ is the algebra of polynomials using those elements as non-commuting independent variables. More explicitly: given a basis $\mathbf{e}_1, ..., \mathbf{e}_d$ of linearly independent elements of $\mathcal{V}$, the tensor algebra $T(\mathcal{V})$ has a basis consisting of the scalar 1, the vectors $\mathbf{e}_i$ (d of these), the products $\mathbf{e}_i\mathbf{e}_j$ (d^2 of these), the products $\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k$ (d^3 of these), and so on. All of these elements of $T(\mathcal{V})$ are linearly independent (so $T(\mathcal{V})$ is infinite-dimensional), and every element of $T(\mathcal{V})$ is a linear combination of these. The scalar 1 serves as the multiplicative unit element. The product is associative, distributive, and linear – and that's all.[3] In particular, $\mathbf{e}_j\mathbf{e}_k$ and $\mathbf{e}_k\mathbf{e}_j$ are not proportional to each other unless $j = k$. The product in this algebra is called the **tensor product**.[5]

[1] This article assumes that d is finite.

[2] The definition of *vector space* is reviewed in Pinter (1990), chapter 28.

[3] This loose idea can be made mathematically precise using the language of **category theory**, where it can be expressed as a **universal property**.

[4] The general definition of *associative algebra* is reviewed in Jacobson (1985), section 7.1, definition 1. The general definition doesn't require having a multiplicative unit, but the tensor algebra does.

[5] The tensor product of A and B is usually denoted $A \otimes B$, but I'm writing it more concisely as AB.

2 The definition of Clifford algebra

At any given point in spacetime, we can use the metric to form a Lorentz-invariant scalar product of two vectors.[6] That's an example of a *symmetric bilinear form*. Given a vector space $\mathcal{V}$, a **symmetric bilinear form** on $\mathcal{V}$ assigns a scalar $g(\mathbf{a},\mathbf{b})$ to each pair of vectors $\{\mathbf{a},\mathbf{b}\}$, subject to these conditions:[7]

$$g(\mathbf{a},\mathbf{b}) = g(\mathbf{b},\mathbf{a}) \qquad\qquad g(s\mathbf{a},\mathbf{b}) = sg(\mathbf{a},\mathbf{b})$$
$$g(\mathbf{a}+\mathbf{b},\mathbf{c}) = g(\mathbf{a},\mathbf{c}) + g(\mathbf{b},\mathbf{c})$$

for all vectors $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathcal{V}$ and all scalars s.

Given a vector space $\mathcal{V}$ and a symmetric bilinear form $g(\cdot,\cdot)$ on $\mathcal{V}$, the corresponding **Clifford algebra** $\mathrm{Cliff}(\mathcal{V},g)$ is defined just like the tensor algebra $T(\mathcal{V})$ but with the additional relationship[8]

$$\mathbf{v}^2 = g(\mathbf{v},\mathbf{v}) \tag{1}$$

for all vectors $\mathbf{v} \in \mathcal{V}$. This means that in any product of vectors where the same vector $\mathbf{v}$ occurs as two consecutive factors, we can replace that pair of factors with the scalar $g(\mathbf{v},\mathbf{v})$. With this modification, the tensor product becomes the **Clifford product**. It still satisfies the generic axioms of associative algebra, but it also satisfies this extra relationship that cannot be derived from those generic axioms.[9]

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[6] Article 48968

[7] The first condition says that it's *symmetric*, and then the other conditions say that it's *linear* in each argument (*bilinear*).

[8] In sophisticated terms: the Clifford algebra is the quotient $T(\mathcal{V})/\mathcal{I}(\mathcal{V},g)$, where $T(\mathcal{V})$ is the tensor algebra and $\mathcal{I}(\mathcal{V},g) \subset T(\mathcal{V})$ is the ideal generated by elements of the form $\mathbf{v}^2 - g(\mathbf{v},\mathbf{v})$ for all vectors $\mathbf{v}$. The **ideal** generated by a subset $\mathcal{S} \subset T(\mathcal{V})$ is the smallest subalgebra containing $\mathcal{S}$ with the property that ab is in the subalgebra whenever either a or b is in the subalgebra, and the quotient $T(\mathcal{V})/\mathcal{I}(\mathcal{V},g)$ means "just like $T(\mathcal{V})$ but with additional relationships saying that every element of $\mathcal{I}(\mathcal{V},g)$ is equivalent to zero."

[9] In the tensor algebra, if the vectors **a** and **b** are linearly independent, then the products **ab** and **ba** are not proportional to each other. But in the Clifford algebra, they are proportional to each other if $g(\mathbf{a},\mathbf{b}) = 0$ (section 3).

3 The size of a Clifford algebra

Equation (1) implies[10]

$$\mathbf{ab} + \mathbf{ba} = 2g(\mathbf{a}, \mathbf{b}) \tag{2}$$

for all vectors $\mathbf{a}, \mathbf{b}$. In particular,

$$\mathbf{ab} = -\mathbf{ba} + \text{scalar},$$

so any product $\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\cdots$ of basis vectors may be rewritten as a linear combination of such products that each involve a strictly increasing sequence of indices. Only 2^d strictly increasing sequences exist, because the index-set is $\{1, 2, ..., d\}$. This shows that the Clifford algebra is only 2^d-dimensional, even though the tensor algebra $T(\mathcal{V})$ is infinite-dimensional.

In physics, Clifford algebras are often defined by choosing a basis $\mathbf{e}_1, ..., \mathbf{e}_d$ for $\mathcal{V}$ and requiring that the basis vectors satisfy equation (2):[11]

$$\mathbf{e}_j\mathbf{e}_k + \mathbf{e}_k\mathbf{e}_j = 2g(\mathbf{e}_j, \mathbf{e}_k).$$

This implies equation (1), because every vector $\mathbf{v}$ is a linear combination of the basis vectors $\mathbf{e}_k$.

[10] To deduce this, use $\mathbf{a}^2 = g(\mathbf{a}, \mathbf{a})$ and $\mathbf{b}^2 = g(\mathbf{b}, \mathbf{b})$ and $(\mathbf{a} + \mathbf{b})^2 = g(\mathbf{a} + \mathbf{b}, \mathbf{a} + \mathbf{b})$.

[11] When a matrix representation is used, so that each element of the Clifford algebra is represented by a square matrix using the matrix product as the Clifford product, a matrix representing one of the basis vectors $\mathbf{e}_k$ is called a **Dirac matrix**, especially when $g(\mathbf{e}_j, \mathbf{e}_k) = \pm 1$ for $j = k$ and $g(\mathbf{e}_j, \mathbf{e}_k) = 0$ otherwise.

4 Degenerate and nongenenerate Clifford algebras

The simplest example of a symmetric bilinear form is the one with $g(\mathbf{v},\mathbf{v}) = 0$ for all vectors $\mathbf{v}$.[12] This is an extreme example of a *degenerate* bilinear form. More generally, a bilinear form g is called **degenerate** if a nonzero vector $\mathbf{a}$ exists for which $g(\mathbf{a},\mathbf{b}) = 0$ for all vectors $\mathbf{b}$.

If a symmetric bilinear form g is **nondegenerate** (not degenerate), then $\mathcal{V}$ has a basis $\mathbf{e}_1, ..., \mathbf{e}_d$ for which $g(\mathbf{e}_j, \mathbf{e}_k)$ is nonzero whenever $j = k$ and zero otherwise. A Clifford algebra corresponding to a nondegenerate bilinear form will be called a **nondegenerate Clifford algebra**.

If the field of scalars is $\mathbb{R}$, then a nondegenerate bilinear form has a **signature** (p, q), where p and q are the numbers of positive and negative values of $g(\mathbf{e}_k, \mathbf{e}_k)$, respectively, in a basis for $\mathcal{V}$ that makes the matrix $M_{jk} \equiv g(\mathbf{e}_j, \mathbf{e}_k)$ diagonal. The signature is called **euclidean** if either p or q is zero, and if both are nonzero then it's called **lorentzian** if either p or q is 1. When the signature is euclidean, $g(\mathbf{a},\mathbf{b})$ is the familiar **dot product** $\mathbf{a}\cdot\mathbf{b}$. When the signature is lorentzian, $g(\mathbf{a},\mathbf{b})$ is the Lorentz-invariant scalar product.[13]

For nondegenerate Clifford algebras, this notation will be used:

- When $\mathcal{V}$ is a d-dimensional vector space over $\mathbb{C}$, $\text{Cliff}(\mathcal{V}, g)$ will be denoted **Cliff(d)**, using a single integer d to indicate the number of dimensions of $\mathcal{V}$.
- When $\mathcal{V}$ is a d-dimensional vector space over $\mathbb{R}$, $\text{Cliff}(\mathcal{V}, g)$ will be denoted **Cliff(p, q)** , using a pair of integers (p, q) to indicate the signature of g.

Clifford algebra is often called **geometric algebra**,[14] at least when the field of scalars is $\mathbb{R}$ and the bilinear form g is nondegenerate.

[12] In this special case, the Clifford algebra $\text{Cliff}(\mathcal{V}, g)$ is the same as the **exterior algebra** that was introduced in article 81674 using a different perspective.

[13] This is a tautology, because Lorentz transformations are *defined* to be those linear transformations of $\mathcal{V}$ that leave $g(\mathbf{a},\mathbf{b})$ invariant for all $\mathbf{a},\mathbf{b} \in \mathcal{V}$.

[14] According to Hestenes and Sobczyk (1992), this name was suggested by Clifford himself.

5 The Clifford product of two vectors

Two vectors $\mathbf{a}, \mathbf{b}$ are called **orthogonal** to each other if $g(\mathbf{a}, \mathbf{b}) = 0$. Equation (2) implies that if two vectors are orthogonal with each other, then they anticommute with each other:

$$\text{if } g(\mathbf{a}, \mathbf{b}) = 0, \text{ then } \mathbf{ab} = -\mathbf{ba}. \tag{3}$$

For any two vectors $\mathbf{a}$ and $\mathbf{b}$, we can always decompose their Clifford product into a symmetric part and an antisymmetric part:

$$\mathbf{ab} = \frac{\mathbf{ab} + \mathbf{ba}}{2} + \frac{\mathbf{ab} - \mathbf{ba}}{2}. \tag{4}$$

Equation (2) says that the symmetric part is the scalar $g(\mathbf{a}, \mathbf{b})$.

The antisymmetric part is denoted $\mathbf{a} \wedge \mathbf{b}$ and called the **wedge product** (or **exterior product**) of $\mathbf{a}$ and $\mathbf{b}$. It is nonzero if and only if $\mathbf{a}$ and $\mathbf{b}$ are linearly independent. Two linearly independent vectors span a plane (a two-dimensional subspace), so the wedge product $\mathbf{a} \wedge \mathbf{b}$ has a natural geometric interpretation using the plane spanned by $\mathbf{a}$ and $\mathbf{b}$. The wedge product is a d-dimensional generalization of the traditional cross-product of two vectors in three-dimensional space,[15] but it is not a vector. It's called a **bivector**. More generally, any linear combination of bivectors is again called a bivector, even if it cannot be written as the wedge product of two vectors (cannot be interpreted geometrically using a single plane).[16,17] A bivector that can be written as $\mathbf{a} \wedge \mathbf{b}$ is called a **simple bivector**.

Neither $g(\mathbf{a}, \mathbf{b})$ nor $\mathbf{a} \wedge \mathbf{b}$ is enough information to determine one of the two vectors from the other one, but if $g(\mathbf{a}, \mathbf{b})$ or $\mathbf{a} \wedge \mathbf{b}$ are both given (so that $\mathbf{ab}$ is known), then the situation is better: given $\mathbf{a}$ and $\mathbf{ab}$, if $g(\mathbf{a}, \mathbf{a}) \neq 0$,[18] the product of $\mathbf{a}/g(\mathbf{a}, \mathbf{a})$ with $\mathbf{ab}$ gives $\mathbf{b}$. More briefly: if $g(\mathbf{a}, \mathbf{a}) \neq 0$, then $\mathbf{a}$ is invertible.

[15] Article 81674

[16] Example: $\mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4$, where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ are mutually orthogonal.

[17] Given an orthogonal basis $\mathbf{e}_1, ..., \mathbf{e}_d$ for $\mathcal{V}$, a bivector is a linear combination of products $\mathbf{e}_j\mathbf{e}_k$ with $j < k$. A linear combination of products $\mathbf{e}_j\mathbf{e}_k\mathbf{e}_\ell$ with $j < k < \ell$ is called a **trivector**.

[18] If $g(\mathbf{a}, \mathbf{a}) = 0$, then we still don't have enough information, because then $\mathbf{ab} = \mathbf{a}(\mathbf{b} + s\mathbf{a})$ for every scalar s.

6 The square of a simple bivector

The quantity $(\mathbf{ab})^2$ is not necessarily a scalar, but the quantity $(\mathbf{a}\wedge\mathbf{b})^2$ is always a scalar. Proof:

$$\begin{aligned}(\mathbf{a}\wedge\mathbf{b})^2 &= \left(\frac{\mathbf{ab}-\mathbf{ba}}{2}\right)^2\\ &= \left(\frac{\mathbf{ab}+\mathbf{ba}}{2}\right)^2 - \frac{\mathbf{abba}+\mathbf{baab}}{2}\\ &= \big(g(\mathbf{a},\mathbf{b})\big)^2 - g(\mathbf{a},\mathbf{a})g(\mathbf{b},\mathbf{b}).\end{aligned}$$

This is true in every Clifford algebra, degenerate or not. It implies

$$(\mathbf{a}\wedge\mathbf{b})^3 \propto \mathbf{a}\wedge\mathbf{b}.$$

In the euclidean case, the quantity $\sqrt{|(\mathbf{a}\wedge\mathbf{b})^2|}$ defines a unit of area associated with the two-dimensional subspace spanned by the vectors $\mathbf{a}$ and $\mathbf{b}$, so $\mathbf{a}\wedge\mathbf{b}$ may be interpreted geometrically as an oriented element of area, just like a vector may be interpreted as an oriented element of length. More precisely, we can think of the two vectors $\mathbf{a}$ and $\mathbf{b}$ as the two sides of an oriented parallelogram with lengths $\sqrt{\mathbf{a}^2}$ and $\sqrt{\mathbf{b}^2}$, and the quantity

$$\sqrt{|(\mathbf{a}\wedge\mathbf{b})^2|} = \sqrt{(\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})^2}$$

is the area of this parallelogram.[19]

The square of a generic (not necessarily simple) bivector is not necessarily a scalar. Example: if $A = \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_4$, where the $\mathbf{e}_k$s are mutually orthogonal vectors, then $A^2 = 2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4 + \text{scalar}$.

[19]The quantity under the square root is the product of the lengths of the vectors $\mathbf{a}$ and $\mathbf{b}'$, where $\mathbf{b}' = \mathbf{b}-(\mathbf{b}\cdot\mathbf{a}/\mathbf{a}\cdot\mathbf{a})\mathbf{a}$ is the component of $\mathbf{b}$ orthogonal to $\mathbf{a}$. This agrees with the familiar "width times height" rule for the area of a parallelogram.

7 A Clifford algebra is more than just an algebra

A **homomorphism** from one algebra A to another algebra B is a map

$$\rho : A \to B$$

for which

$$\rho(a + a') = \rho(a) + \rho(a') \qquad\qquad \rho(sa) = s\rho(a)$$
$$\rho(aa') = \rho(a)\rho(a')$$

for all $a, a' \in A$ and all scalars s. If another homomorphism

$$\rho^{-1} : B \to A$$

exists with

$$\rho^{-1}\big(\rho(a)\big) = a \qquad\qquad \rho\big(\rho^{-1}(b)\big) = b$$

for all $a \in A$ and all $b \in B$, then the homomorphism is called an **isomorphism**, and A and B are said to be **isomorphic** to each other as algebras, denoted $A \simeq B$. In colloquial terms: A and B are isomorphic to each other if they are the same as algebras, even if they differ from each other in other ways.

Two Clifford algebras may be the same (isomorphic) as algebras even if they are different as Clifford algebras, because the definition of a Clifford algebra involves more than just its algebraic structure: the data that specifies a Clifford algebra $\text{Cliff}(\mathcal{V}, g)$ includes $\mathcal{V}$ and g, which cannot always be inferred from the algebraic structure of $\text{Cliff}(\mathcal{V}, g)$. The next section demonstrates this.

8 The signature cannot be inferred from the algebra

The Clifford algebras Cliff$(p+1,q)$ and Cliff$(q+1,p)$ differ from each other as Clifford algebras (when $p\neq q$), but they are the same as algebras:[20,21]

$$\text{Cliff}(p+1,q)\simeq\text{Cliff}(q+1,p). \tag{5}$$

This shows that the signature of a Clifford algebra typically cannot be inferred from its algebraic structure alone.

To deduce the isomorphism (5), let

$$\mathbf{h},\ \mathbf{e}_1,\ ...,\ \mathbf{e}_p,\ \mathbf{f}_1,\ ...,\ \mathbf{f}_q$$

be a set of mutually orthogonal vectors in Cliff$(p+1,q)$ satisfying

$$\mathbf{h}^2=1 \qquad\qquad \mathbf{e}_k^2=1 \qquad\qquad \mathbf{f}_k^2=-1.$$

Now define

$$\mathbf{e}'_k\equiv\mathbf{h}\mathbf{e}_k \qquad\qquad \mathbf{f}'_k\equiv\mathbf{h}\mathbf{f}_k.$$

The quantities

$$\mathbf{h},\ \mathbf{f}'_1,\ ...,\ \mathbf{f}'_q,\ \mathbf{e}'_1,\ ...,\ \mathbf{e}'_p$$

all anti-commute with each other ($\mathbf{ab}=-\mathbf{ba}$), and they satisfy

$$(\mathbf{f}'_k)^2=1 \qquad\qquad (\mathbf{e}'_k)^2=-1,$$

so they have all the right algebraic properties to be re-interpreted as a set of mutually orthogonal vectors generating Cliff$(q+1,p)$. This shows that Cliff$(p+1,q)$ and Cliff$(q+1,p)$ are the same (isomorphic) as algebras. They differ from each other as Clifford algebras, because $\mathbf{e}'_k$ and $\mathbf{f}'_k$ are designated as *vectors* in Cliff$(q+1,p)$ but as *bivectors* in Cliff$(p+1,q)$, and plain algebra ignores those designations.

[20] Benn and Tucker (1989), page 32, equation (2.2.7)

[21] The notation Cliff(p,q) was defined in section 4.

9 Isomorphisms involving the even subalgebra

If A is an algebra, then a subset $B \subset A$ is called a **subalgebra** if all linear combinations and products of elements of B are also elements of B. Every Clifford algebra $\text{Cliff}(\mathcal{V}, g)$ has a subalgebra called the **even subalgebra**, which is the smallest subalgebra containing all products of vectors with an even number of vectors in each product. The even subalgebra of $\text{Cliff}(\mathcal{V}, g)$ will be denoted $\text{Cliff}_{\text{even}}(\mathcal{V}, g)$.

Each nondegenerate Clifford algebra is isomorphic (as an algebra) to the even subalgebra of a Clifford algebra in a higher dimension. For nondegenerate Clifford algebras over $\mathbb{C}$, the relationship is[22,23]

$$\text{Cliff}(d) \simeq \text{Cliff}_{\text{even}}(d+1).$$

For nondegenerate Clifford algebras over $\mathbb{R}$, the relationship is[24]

$$\text{Cliff}(q, p) \simeq \text{Cliff}_{\text{even}}(p+1, q). \tag{6}$$

Notice that p and q switch positions.[25] Example: $\text{Cliff}(1, 0) \simeq \text{Cliff}_{\text{even}}(1, 1)$.

The relationship (6) can be deduced using the same approach that was used in section 8. Let

$$\mathbf{h},\ \mathbf{e}_1,\ ...,\ \mathbf{e}_p,\ \mathbf{f}_1,\ ...,\ \mathbf{f}_q$$

be a set of mutually orthogonal vectors in $\text{Cliff}(p+1, q)$ satisfying

$$\mathbf{h}^2 = 1 \qquad\qquad \mathbf{e}_k^2 = 1 \qquad\qquad \mathbf{f}_k^2 = -1.$$

Now define $\mathbf{e}'_k \equiv \mathbf{h}\mathbf{e}_k$ and $\mathbf{f}'_k \equiv \mathbf{h}\mathbf{f}_k$, just like in section 8. These quantities generate the even subalgebra of $\text{Cliff}(p+1, q)$. They are designated as *bivectors* in the context of $\text{Cliff}(p+1, q)$, but their algebraic properties are consistent with re-interpreting them as mutually orthogonal *vectors* in $\text{Cliff}(q, p)$.

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[22] Benn and Tucker (1989), page 80, equation (2.7.5)

[23] The notation $\text{Cliff}(d)$ was defined in section 4.

[24] Benn and Tucker (1989), page 39, equation (2.3.1)

[25] The relationship $\text{Cliff}_{\text{even}}(p, q) \simeq \text{Cliff}_{\text{even}}(q, p)$ also holds. To check this informally, use the fact that replacing every vector $\mathbf{v} \in \text{Cliff}(p, q)$ with $i\mathbf{v}$ (with $i^2 = -1$) converts $\text{Cliff}(p, q)$ to $\text{Cliff}(q, p)$.

10 The algebraic structure of Clifford algebras

The concept of isomorphism defined in section 7 can be used to relate the algebraic structure of a Clifford algebra to that of another standard algebra. This section summarizes how the nondegenerate Clifford algebras are related to matrix algebras.

For nondegenerate Clifford algebras over $\mathbb{C}$, the relationships follow a relatively simple pattern:[26]

$$
\begin{aligned}
\text{Cliff}(2n) &\simeq M_{\mathbb{C}}(2^n) \\
\text{Cliff}(2n+1) &\simeq M_{\mathbb{C}}(2^n) \oplus M_{\mathbb{C}}(2^n),
\end{aligned}
$$

where $M_{\mathbb{C}}(k)$ is the algebra of $k \times k$ matrices over $\mathbb{C}$.

For nondegenerate Clifford algebras over $\mathbb{R}$, the relationships follow a more elaborate pattern. Let $M_{\mathbb{R}}(k)$ be the algebra of $k \times k$ matrices over $\mathbb{R}$, and let $M_{\mathbb{H}}(k)$ be the algebra of $k \times k$ matrices whose components are elements of the quaternion algebra[27] $\mathbb{H}$. For each p, q, an integer $k > 0$ exists for which these relationships hold:[28,29,30]

$p - q$ modulo 8	$\text{Cliff}(p,q) \simeq$
0	$M_{\mathbb{R}}(k)$
1	$M_{\mathbb{R}}(k) \oplus M_{\mathbb{R}}(k)$
2	$M_{\mathbb{R}}(k)$
3	$M_{\mathbb{C}}(k)$

$p - q$ modulo 8	$\text{Cliff}(p,q) \simeq$
4	$M_{\mathbb{H}}(k)$
5	$M_{\mathbb{H}}(k) \oplus M_{\mathbb{H}}(k)$
6	$M_{\mathbb{H}}(k)$
7	$M_{\mathbb{C}}(k)$

[26]Figueroa-O'Farrill (2015), section 4, table 3. Beware of the error in Benn and Tucker (1989)'s equation (2.7.4b).

[27]The **quaternion algebra** may itself be defined using the isomorphism $\mathbb{H} \simeq \text{Cliff}(0,2)$, which is one of the cases listed in the table. Thanks to the isomorphism $\text{Cliff}(0,2) \simeq \text{Cliff}_{\text{even}}(3,0)$ which is a special case of (6), the quaternion algebra may also be defined using $\mathbb{H} \simeq \text{Cliff}_{\text{even}}(3,0)$. Explicitly: if $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is a basis of orthogonal vectors in $\text{Cliff}(3,0)$, then the three bivectors $\mathbf{i} \equiv \mathbf{e}_1\mathbf{e}_2$ and $\mathbf{j} \equiv \mathbf{e}_2\mathbf{e}_3$ and $\mathbf{k} \equiv \mathbf{e}_3\mathbf{e}_1$ generate $\mathbb{H}$.

[28]Benn and Tucker (1989), page 35, equation (2.2.10), and page 40, table 2.8. Sometimes the Clifford algebra is defined with an extra minus sign, using $\mathbf{v}^2 = -g(\mathbf{v}, \mathbf{v})$ instead of equation (1). Example: Figueroa-O'Farrill (2015). That switches the roles of p and q in the table.

[29]In these relationships, the matrix algebras are regarded as algebras over $\mathbb{R}$, even though their components may belong to $\mathbb{C}$ or $\mathbb{H}$. In other words, two matrices X_1 and X_2 are considered to be linearly independent if $r_1X_1 + r_2X_2 \neq 0$ for all nonzero real numbers r_1, r_2.

[30]The isomorphisms (5) can be inferred from this table.

11 Examples, part 1

Consider Cliff$(0,1)$, and let $\mathbf{i} \in$ Cliff$(0,1)$ be a vector satisfying $\mathbf{i}^2 = -1$. All elements of this Clifford algebra have the form $x + y\mathbf{i}$, where x, y are arbitrary real numbers. This shows that Cliff$(0,1)$ is isomorphic to the algebra of complex numbers:

$$\text{Cliff}(0,1) \simeq \mathbb{C}. \tag{7}$$

Use

$$0 - 1 = 7 \qquad\qquad (\text{modulo } 8)$$

to see that this is consistent with the table in section 10, with $k = 1$ in this case. In the relationship (7), $\mathbb{C}$ is interpreted as an algebra over $\mathbb{R}$, so that the real and imaginary units are linearly independent.

Next, consider Cliff$(1,0)$, and let $\mathbf{v} \in$ Cliff$(1,0)$ be a vector satisfying $\mathbf{v}^2 = 1$. Every element of this Clifford algebra has the form $x + y\mathbf{v}$, where x, y are arbitrary real numbers. Alternatively, every element of Cliff$(1,0)$ may be written

$$x_+P_+ + x_-P_-$$

with $P_\pm \equiv (1 + \mathbf{v})/2$, where $x_\pm$ are arbitrary real numbers. The quantities $P_\pm$ commute with everything (including each other), and they satisfy $P_\pm^2 = P_\pm$, so

$$\text{Cliff}(1,0) \simeq \mathbb{R} \oplus \mathbb{R}. \tag{8}$$

This is consistent with the table in section 10, with $k = 1$ in this case. In the relationship (8), the first $\mathbb{R}$ in $\mathbb{R} \oplus \mathbb{R}$ is the one-dimensional algebra generated by P_+ with coefficients in the field of real numbers (using P_+ as the identity element), and the second $\mathbb{R}$ in $\mathbb{R} \oplus \mathbb{R}$ is the one-dimensional algebra generated by P_- with coefficients in the field of real numbers (using P_- as the identity element).

In section 10, the cases involving "$\oplus$" are precisely the cases for which Cliff(p,q) has a set of mutually orthogonal vectors $\mathbf{e}_1, ..., \mathbf{e}_{p+q}$ whose product $Z \equiv \mathbf{e}_1 \cdots \mathbf{e}_{p+q}$ commutes with everything and satisfies $Z^2 = 1$. Just like in the preceding example, we can use this to define two mutually orthogonal projection operators $P_\pm \equiv (1 \pm Z)/2$ that commute with everything. This explains the "$\oplus$" structure.

12 Examples, part 2

Now consider Cliff(1, 1). Let $\mathbf{e}, \mathbf{f} \in \text{Cliff}(1,1)$ be mutually orthogonal vectors satisfying $\mathbf{e}^2 = 1$ and $\mathbf{f}^2 = -1$. Every element of this Clifford algebra is a linear combination of 1, $\mathbf{e}$, $\mathbf{f}$, and $\mathbf{ef}$, with real numbers as coefficients. If we use the matrix representation

$$1 \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{e} \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{f} \to \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \mathbf{ef} \to \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

then the matrix product satisfies the same multiplication table as the Clifford product. This shows that

$$\text{Cliff}(1,1) \simeq M_{\mathbb{R}}(2),$$

which is consistent with the table in section 10, with $k = 2$ in this case.

Next, consider Cliff(4, 1) and Cliff(4). These are different from each other as Clifford algebras, and they are also different from each other as plain algebras, because Cliff(4, 1) uses the real numbers $\mathbb{R}$ as its field of scalars, whereas Cliff(4) uses the complex numbers $\mathbb{C}$ as its field of scalars. However, we can also think of Cliff(4) as an algebra over $\mathbb{R}$, meaning we think of two elements $X, Y \in \text{Cliff}(4)$ as being linearly independent if $xX + yY \neq 0$ for all nonzero real numbers x, y. When we think of Cliff(4) this way, then

$$\text{Cliff}(4,1) \simeq \text{Cliff}(4). \tag{9}$$

This is consistent with the relationships in section 10, namely $\text{Cliff}(4,1) \simeq M_{\mathbb{C}}(2)$ and $\text{Cliff}(4) \simeq M_{\mathbb{C}}(2)$, if we think of all of these algebras as algebras over $\mathbb{R}$. To deduce the relationship (9), let $\mathbf{e}_1, ..., \mathbf{e}_4$ and $\mathbf{f}$ be a set of mutually orthogonal vectors in Cliff(4, 1) with $\mathbf{e}_k^2 = 1$ and $\mathbf{f}^2 = -1$. Define $\mathbf{i} \equiv \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4\mathbf{f}$, which satisfies $\mathbf{i}^2 = -1$ and commutes with everything in Cliff(4, 1). The relationship (9) follows from the fact that every element of Cliff(4, 1) may be written as a linear combination of the $\mathbf{e}_k$ with coefficients of the form $x + y\mathbf{i}$ with $x, y \in \mathbb{R}$.

13 Examples, part 3

As a final example, consider Cliff(5,0). According to the table in section 10,

$$\text{Cliff}(5,0) \simeq M_{\mathbb{H}}(2) \oplus M_{\mathbb{H}}(2). \tag{10}$$

To deduce this, let $\mathbf{e}_1, ..., \mathbf{e}_5$ be mutually orthogonal vectors in Cliff(5,0) satisfying $\mathbf{e}_k^2 = 1$. Let $C_4 \subset \text{Cliff}(5,0)$ be the subalgebra generated by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$, which is clearly isomorphic to Cliff(4,0). Let $H \subset C_4$ be the subalgebra consisting of elements of the form

$$\mathbf{h} = w + x\mathbf{e}_2\mathbf{e}_3 + y\mathbf{e}_3\mathbf{e}_1 + z\mathbf{e}_1\mathbf{e}_2 \qquad\qquad w, x, y, z \in \mathbb{R}$$

so that H is isomorphic to the quaternion algebra $\mathbb{H}$ (footnote 27 in section 10). The quantities $\mathbf{e}_4$ and $\mathbf{f} \equiv \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ anticommute with each other, commute with everything in H, and satisfy $\mathbf{e}_4^2 = 1$ and $\mathbf{f}^2 = -1$. This can be used to show that if we use the matrix representation

$$\mathbf{h}\mathbf{e}_4 = \begin{bmatrix} 0 & \mathbf{h} \\ \mathbf{h} & 0 \end{bmatrix} \qquad\qquad \mathbf{h}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = \begin{bmatrix} 0 & \mathbf{h} \\ -\mathbf{h} & 0 \end{bmatrix}, \tag{11}$$

then the matrix product agrees with the Clifford product. The quantities $\mathbf{e}_4$ and $\mathbf{f}$ together with H generate all of C_4, and matrices of the form (11) generate all of $M_H(2)$ (the algebra of 2×2 matrices over H),[31] so

$$C_4 \simeq M_{\mathbb{H}}(2). \tag{12}$$

The product $Z \equiv \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4\mathbf{e}_5$. commutes with everything and satisfies $Z^2 = 1$, so the quantities $P_\pm \equiv (1 \pm Z)/2$ commute with everything (including each other) and satisfy $P_\pm^2 = P_\pm$ and $P_+P_- = 0$. The subsets C_4P_+ and C_4P_- are mutually commuting subalgebras (with unit elements P_+ and P_-, respectively) that are both isomorphic to C_4 and together span Cliff(5,0), so we have deduced $\text{Cliff}(5,0) = C_4P_+ \oplus C_4P_- \simeq C_4 \oplus C_4$. Combine this with (12) to get the result (10).

[31] Notice the font: H versus $\mathbb{H}$. Compare to equation (12).

14 References

Benn and Tucker, 1989. *An Introduction to Spinors and Geometry with Applications in Physics.* Adam Hilgar

Figueroa-O'Farrill, 2015. "Majorana spinors" https://www.maths.ed.ac.uk/~jmf/Teaching/Lectures/Majorana.pdf

Hestenes and Sobczyk, 1992. *Clifford Algebra to Geometric Calculus.* D. Reidel Publishing Company

Jacobson, 1985. *Basic Algebra I (Second Edition).* Dover (2009)

Pinter, 1990. *A Book of Abstract Algebra (Second Edition).* Dover (2010)

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