The Hopf Bundle: an Example of a Nontrivial Principal Bundle

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Abstract The **Hopf bundle**, also called the **Hopf fibration**, is a relatively simple example of a principal bundle. A **principal bundle** is a mathematical structure that is important in the study of gauge fields. Articles 70621 and 76708 introduced some of the general concepts, and this article uses the Hopf bundle to illustrate a few of them in detail.

Contents

1 The Hopf bundle

To illustrate some of the concepts that were defined in articles 70621 and 76708, this article describes a nontrivial principal $U(1)$-bundle called the **Hopf bundle** or the **Hopf fibration**. Its structure group $U(1)$ is the group of complex numbers of magnitude 1 with ordinary multiplication as the product. The total space is S^3 and the base space is S^2, where S^k denotes the k-dimensional **sphere** – the set of points in $k+1$-dimensional euclidean space whose distance from the origin is 1.

The total space S^3 and the base space S^2 will both be described using a pair of complex variables z, w. To describe the total space S^3, use the fact that the real and imaginary parts of z, w are coordinates for a four-dimensional space $\mathbb{R}^4$. Then the condition

$$|z|^2 + |w|^2 = 1 \tag{1}$$

describes a three-dimensional manifold S^3 embedded in $\mathbb{R}^4$. Each point of S^3 is a pair (z, w) satisfying the condition (1). The action of the structure group $U(1)$ on the total space S^3 is defined by[1,2]

$$(z, w) \times \lambda = (\lambda z, \lambda w)$$

with $\lambda \in U(1)$. To describe the base space S^2, consider the equivalence relation[3]

$$(\lambda z, \lambda w) \sim (z, w) \tag{2}$$

for all $\lambda \in U(1)$. Let $[z, w]$ denote[4] the equivalence class that includes the pair (z, w), and define the bundle projection $p : S^3 \to S^2$ by

$$p(z, w) = [z, w]. \tag{3}$$

[1] Wendl (2007), example 2.92

[2] Since the structure group is abelian in this example, we don't need to distinguish between the left and right actions of the structure group.

[3] Nakahara (1990), text around equation (9.58)

[4] Articles 70621 and 76708 used the notation $[\cdot, \cdot]$ for other things, not related to the way it's used here.

To see that the image of this projection really is S^2, cover S^2 with two charts, each of which excludes only one point of S^2: one chart $U_{w\neq 0}$ that excludes the point $[z, w]$ with $w = 0$, and one chart $U_{z\neq 0}$ that excludes the point $[z, w]$ with $z = 0$. Use the real and imaginary parts of $x + iy \equiv z/w$ as coordinates for $U_{w\neq 0}$, and use the real and imaginary parts of $x' + iy' \equiv w/z$ as coordinates for $U_{z\neq 0}$. These two charts are glued together by the relation $x' + iy' = 1/(x + iy)$ wherever z and w are both nonzero. For the chart $U_{w\neq 0}$, the map

$$(z, w) \mapsto \big(z/w,\ w/|w|\big) \tag{4}$$

is a local trivialization, with inverse[5]

$$(a, b) \mapsto (ab/c,\ b/c) \qquad\qquad c \equiv \sqrt{|a|^2 + |b|^2}.$$

For the chart $U_{z\neq 0}$, the map

$$(z, w) \mapsto \big(w/z,\ z/|z|\big) \tag{5}$$

is a local trivialization, with inverse

$$(a, b) \mapsto (b/c,\ ab/c) \qquad\qquad c \equiv \sqrt{|a|^2 + |b|^2}.$$

[5]This assumes that z, w satisfy the condition (1).

2 Comparison to other $U(1)$-bundles over S^2

The Hopf bundle is one example of a principal $U(1)$-bundle over S^2, but it's not the only one. Other examples include:

- The trivial bundle with total space $S^2 \times U(1)$.
- The unit tangent bundle of S^2, obtained from the tangent bundle of S^2 by keeping only the tangent vectors whose length is 1. Article 70621 shows that the total space of this bundle is the real projective space $\mathbb{R}P^3$, so it is distinct from the trivial bundle and from the Hopf bundle.
- The Hopf bundle and the preceding examples all fit neatly into a larger pattern. Oriented principal $U(1)$-bundles over S^2 are uniquely classified (up to equivalence) by an integer n. Such a bundle exists for each integer value of n, including these special cases:
  - The case $n = 0$ is the trivial bundle, with total space $S^2 \times S^1$.
  - The case $n = 1$ is the Hopf bundle, with total space S^3.
  - The case $n = 2$ is the unit tangent bundle of S^2, with total space $\mathbb{R}P^3$.

  Chapters 21-25 will describe the generalization to arbitrary n in more detail.

3 The Lie algebra

In this article, the structure group is the Lie group $U(1)$, which we can think of as a one-dimensional manifold consisting of all complex numbers of the form $e^{i\phi}$ with ordinary multiplication as the group operation. The Lie algebra is one-dimensional, spanned by a single element. We can think of elements of a Lie algebra in a few different ways:[6]

- An element of the Lie algebra is a right-invariant vector field on the Lie-group manifold.[7] With this perspective, the Lie algebra of $U(1)$ is spanned by the single vector field ∂_ϕ.
- An element of the Lie algebra is a vector tangent to the Lie-group manifold at the point corresponding to the identity element.[8] With this perspective, the Lie algebra of $U(1)$ is spanned by the single tangent vector $\partial_\phi\big|_{\phi=0}$.
- For a matrix group, an element of the Lie algebra is a matrix, which generates a one-parameter subgroup of the Lie group through exponentiation.[9] With this perspective, the Lie algebra of $U(1)$ is spanned by the "matrix" with only one row and one column, whose entry is the imaginary unit i.[10]

We can relate the first perspective to the third one by using the identity

$$e^{\theta\partial_\phi}e^{i\phi} = e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}.$$

The first equality says that the vector field ∂_ϕ generates translations on the group manifold $U(1)$, and the second equality says that i generates the same translation, now represented as a group operation (multiplication by $e^{i\theta}$). By comparing the translations amounts on both sides of this identity, we learn that the same element of the Lie algebra may be represented either by the vector field ∂_ϕ or by i.

[6] Article 76708

[7] Nakahara (1990), definition 5.48

[8] Fulton and Harris (1991), section 8.1

[9] Article 18505

[10] The Lie algebra of $U(N)$ is naturally generated by antihermitian matrices of size $N\times N$. Sometimes the generators are normalized to be hermitian instead, as in equation (19).

4 Vector fields tangent to the total space

Define real variables x_k by

$$z = x_1 + ix_2 \qquad\qquad w = x_3 + ix_4.$$

Then the condition (1) is

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1. \tag{6}$$

The partial derivatives

$$\partial_k \equiv \frac{\partial}{\partial x_k}$$

are vector fields in the four-dimensional space covered by the coordinates x_k. Each of the combinations

$$L_{jk} \equiv x_j \partial_k - x_k \partial_j$$

annihilates the left-hand side of (6), so the vector fields L_{jk} are all tangent to the manifold S^3 defined by (6).[11] Chapter 5 will show that the three combinations

$$X_V \equiv L_{12} + L_{34} \qquad X_3 \equiv L_{13} + L_{42} \qquad X_4 \equiv L_{14} + L_{23} \tag{7}$$

are linearly independent everywhere on S^3, so they span the tangent space at each point of S^3. The reason for writing the first subscript as V will become clear soon.

We're using four coordinates $x_1, ..., x_4$ to describe a three-dimensional manifold, namely the manifold S^3 defined by the constraint (6). This has the advantage of making some of the patterns more clear,[12] but it also requires some extra care: we need to remember that the coordinates $x_1, ..., x_4$ are not completely independent of each other on the manifold of interest.

[11] If a vector field were not tangent to this S^3, then it would be tangent to a curve along which the value of the left-hand side of (6) is not constant, so it would not annihilate the left-hand side of (6).

[12] This is a common theme in theoretical physics: many patterns can be made more evident by using redundant variables. Article 12883 also illustrates this theme.

5 Linear independence

To prove that the three vector fields (7) are linearly independent everywhere on S^3, suppose that constants a, b, c existed for which

$$aX_V + bX_3 + cX_4 = 0 \tag{8}$$

at some point of S^3. This condition may be written in matrix form like this:

$$C^T \Gamma \begin{bmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \\ \partial_4 \end{bmatrix} = 0 \qquad \text{with } C \equiv \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ and } \Gamma \equiv \begin{bmatrix} -x_2 & x_1 & -x_4 & x_3 \\ -x_3 & x_4 & x_1 & -x_2 \\ -x_4 & -x_3 & x_2 & x_1 \end{bmatrix}.$$

This implies $\Gamma^T C = 0$, which in turn implies $\Gamma\Gamma^T C = 0$, but equation (6) implies that $\Gamma\Gamma^T$ is the identity matrix, so $C = 0$. This shows that at any given point of S^3, the only solution of equation (8) is $a = b = c = 0$, so three vector fields (7) are linearly independent everywhere on S^3.

6 The vertical subspace

The tangent space at each point of a fiber bundle has a distinguished subspace called the **vertical subspace**, which is the subspace tangent to the fiber.[13] This chapter describes the vertical subspace at each point of the Hopf bundle.

The first step is to show that the vector field X_V that was defined in chapter 4 is vertical. To do this, write $\bar{z}$ for the complex conjugate of z. Use the abbreviations

$$\partial_z = \frac{1}{2}\left(\partial_1 - i\partial_2\right) \qquad\qquad \partial_w = \frac{1}{2}\left(\partial_3 - i\partial_4\right)$$

so that $\partial_z z = 1$ and $\partial_z \bar{z} = 0$, and write $\partial_{\bar{z}}$ for the complex conjugate of ∂_z. Then the vector field X_V defined in equation (7) may be written

$$X_V = i(z\partial_z + w\partial_w - \bar{z}\partial_{\bar{z}} - \bar{w}\partial_{\bar{w}}). \tag{9}$$

Equation (3) implies that vertical vectors should annihilate any function of z, w that is invariant under $(z, w) \mapsto (\lambda z, \lambda w)$ whenever $|\lambda| = 1$. Any such function may be written $f(\bar{z}w, \bar{w}z, \bar{z}z, \bar{w}w)$. The vector field (9) annihilates all such functions, so X_V is vertical.

The vertical subspace at each point is one-dimensional because the fiber is a one-dimensional manifold (namely S^1), and X_V is nonzero everywhere on S^3, so the vertical subspace at each point of S^3 is spanned by the vector that the field X_V assigns to that point. This is a way of describing the vertical subspace at each point of the Hopf bundle.

[13] Article 76708

7 Why the vector field X_V is special

The vector field X_V is not the only vertical vector field: multiplying X_V by any scalar function gives another vertical vector field. The vector field X_V is special, though. This chapter explains why.

In a principal G-bundle, the space of vectors tangent to the fiber may be naturally identified with the Lie algebra of G.[14] In this example, the group G is $U(1)$, which we can view as the multiplicative group of complex numbers of the form $e^{i\phi}$ with real ϕ. The Lie algebra is one-dimensional, generated by a single vector field ∂_ϕ that is invariant under the action of the group.[15] Equation (9) shows that the identity

$$\left[\partial_\phi f(e^{i\phi}z, e^{i\phi}w)\right]_{\phi=0} = X_V f(z,w)$$

holds for every function f of the variables z, w and their complex conjugates (or of the real variables x_k defined in chapter 4). This identity shows that when we re-interpret the vertical vector field X_V as a Lie-algebra-valued field, it assigns the same element of the Lie algebra (namely ∂_ϕ) to every point of every fiber. A vertical vector field with this property is called a **fundamental vector field**.[14] The only other vector fields with this property are proportional to X_V with a constant proportionality factor.[16]

[14] Article 76708.

[15] Chapter 3

[16] Multiplying X_V by a non-constant scalar function gives another vertical vector field, but not one that corresponds to the same element of the Lie algebra everywhere.

8 Example of a principal connection

A **connection** projects the tangent space at each point of the total space onto its vertical subspace. The subspaces that get projected to zero are called **horizontal subspaces**.[17] The Hopf bundle admits infinitely many different connections. This chapter describes one of them, which will be used throughout the rest of this article.

The tangent space at each point of S^3 is three-dimensional, and the vertical subspace is one-dimensional, so the horizontal subspace must be two-dimensional. The connection that will be used for the rest of this article is the one for which the vector fields X_3 and X_4 defined in chapter 6 are horizontal.

This connection is a **principal connection**, which means it satisfies an extra condition that is natural to require in the context of a principal bundle: under the action $x \to x\lambda$ of $\lambda \in U(1)$ on $x \in S^3$, the horizontal subspaces remain horizontal. To check that the subspaces spanned by X_3 and X_4 have this property, write them as

$$X_3 = X_1 + X_2 \qquad\qquad X_4 = (X_2 - X_1)/i$$

with

$$X_1 \equiv \bar{z}\partial_w - \bar{w}\partial_z \qquad\qquad X_2 \equiv z\partial_{\bar{w}} - w\partial_{\bar{z}}.$$

When the action $x \to x\lambda$, the vector fields X_1 and X_2 become $\bar{\lambda}^2 X_1$ and $\lambda^2 X_2$, which are still horizontal (still linear combinations of X_3 and X_4 at each point).

[17] Article 76708

9 One-forms and redundant coordinates

In the preceding sections, the three-dimensional manifold S^3 is described using four coordinates $x_1, ..., x_4$ subject to the constraint (6). The differential (or exterior derivative) of equation (6) is

$$x_1\, dx_1 + x_2\, dx_2 + x_3\, dx_3 + x_4\, dx_4 = 0, \tag{10}$$

which says that the one-forms dx_k are not linearly independent. This will be important in chapter 16.

10 Connection one-form

This chapter constructs a one-form ω with these properties:

- $\omega(X) = 0$ whenever X is horizontal according to the connection that was defined in chapter 8,
- $\omega(X_V) = i$.

The imaginary unit i is one way to represent the element of the Lie algebra that the fundamental vector field X_V assigns to every point of the total space,[18] so the one-form ω is the **connection one-form** representing the connection that was defined in chapter 8.

The one-form with these properties is

$$\omega = i(\omega_{12} + \omega_{34}) \tag{11}$$

with

$$\omega_{12} \equiv x_1\,dx_2 - x_2\,dx_1 \qquad\qquad \omega_{34} \equiv x_3\,dx_4 - x_4\,dx_3.$$

To confirm that this has the required properties, use

$$\begin{aligned}
\omega_{12}(L_{12}) &= x_1^2 + x_2^2 & \omega_{12}(L_{13} + L_{42}) &= x_1x_4 + x_2x_3 \\
\omega_{34}(L_{34}) &= x_3^2 + x_4^2 & \omega_{34}(L_{13} + L_{42}) &= -x_1x_4 - x_2x_3 \\
\omega_{12}(L_{34}) &= 0 & \omega_{12}(L_{14} + L_{23}) &= x_2x_4 - x_1x_3 \\
\omega_{34}(L_{12}) &= 0 & \omega_{34}(L_{14} + L_{23}) &= -x_2x_4 + x_1x_3
\end{aligned}$$

together with equation (6).

Instead of using the Lie-algebra-valued one-form ω, the connection may also be represented as a tensor field of type $\binom{1}{1}$,[19] namely $\Phi = X_V \otimes (\omega_{12} + \omega_{34})$. This satisfies $\Phi(X) = 0$ whenever X is horizontal, and $\Phi(X_V) = X_V$.

[18] In chapter 7, the same element of the Lie algebra was represented as ∂_ϕ. Chapter 3 reviewed the relationship between different ways of representing elements of the Lie algebra.

[19] Article 76708

11 Examples of local sections

The Hopf bundle does not have any globally-defined section,[20] but it does have local sections defined in sufficiently small charts. This chapter describes two examples, and chapter 12 will show that they are related to each other by a gauge transformation where they overlap. The notation from chapters 1-4 will be used again here.

First consider the chart defined by $w \neq 0$. In this chart, use coordinates

$$x + iy \equiv \frac{z}{w} = \frac{x_1 + ix_2}{x_3 + ix_4}. \tag{12}$$

Use the abbreviation $\rho \equiv \sqrt{x^2 + y^2}$, and define a section σ by taking $\sigma(x, y)$ to be the point in S^3 with coordinates $(x_1, ..., x_4) = \big(\sigma_1(x, y), ..., \sigma_4(x, y)\big)$ given by

$$\sigma_1(x, y) + i\sigma_2(x, y) = \frac{x + iy}{\sqrt{1 + \rho^2}} \qquad \sigma_3(x, y) + i\sigma_4(x, y) = \frac{1}{\sqrt{1 + \rho^2}}. \tag{13}$$

This is consistent with (12) and with (1).

To cover the point $w = 0$, we can use a different chart, say the chart defined by $z \neq 0$. In this chart, we can use coordinates

$$\tilde{x} + i\tilde{y} \equiv \frac{w}{z} = \frac{x_3 + ix_4}{x_1 + ix_2}, \tag{14}$$

and we can define a section $\tilde{\sigma}$ on this chart by $(x_1, ..., x_4) = \big(\tilde{\sigma}_1(x, y), ..., \tilde{\sigma}_4(x, y)\big)$ with

$$\tilde{\sigma}_1(\tilde{x}, \tilde{y}) + i\tilde{\sigma}_2(\tilde{x}, \tilde{y}) = \frac{1}{\sqrt{1 + \tilde{\rho}^2}} \qquad \tilde{\sigma}_3(\tilde{x}, \tilde{y}) + i\tilde{\sigma}_4(\tilde{x}, \tilde{y}) = \frac{\tilde{x} + i\tilde{y}}{\sqrt{1 + \tilde{\rho}^2}} \tag{15}$$

and $\tilde{\rho} \equiv \sqrt{\tilde{x}^2 + \tilde{y}^2}$.

[20] A principal bundle cannot have a (globally defined) section unless it is trivial (article 70621).

12 Relationship between the two local sections

Where the two charts defined in chapter 11 intersect, which is where z and w are both nonzero, the two coordinate systems for M are related to each other by

$$\tilde{x} + i\tilde{y} = 1/(x + iy)$$

(because $z/w = 1/(w/z)$), which implies

$$\tilde{x} = \frac{x}{\rho^2} \qquad\qquad \tilde{y} = \frac{-y}{\rho^2}. \tag{16}$$

Use this in (15) to write the section $\tilde{\sigma}$ in terms of the coordinates x, y:

$$\tilde{\sigma}_1 + i\tilde{\sigma}_2 = \frac{\rho}{\sqrt{1+\rho^2}} \qquad\qquad \tilde{\sigma}_3 + i\tilde{\sigma}_4 = \frac{x - iy}{\rho\sqrt{1+\rho^2}}. \tag{17}$$

Define ϕ by $x + iy = \rho e^{i\phi}$ to see that the sections (13) and (17) are related to each other by a gauge transformation:

$$\tilde{\sigma}_1 + i\tilde{\sigma}_2 = e^{-i\phi}(\sigma_1 + i\sigma_2) \qquad\qquad \tilde{\sigma}_3 + i\tilde{\sigma}_4 = e^{-i\phi}(\sigma_3 + i\sigma_4) \tag{18}$$

13 Local potentials: two conventions

A connection one-form ω is a Lie-algebra-valued one-form ω defined on the total space. A **local potential**[21] is the pullback of a connection one-form ω by a local section σ. The local section is defined on a chart $U \subset M$, so the resulting local potential is a Lie-algebra-valued one-form on that part of the base space.

When G is a matrix group, an element of the Lie algebra of G is naturally represented as an antisymmetric matrix. Article 76708 defines the local potential as $A \equiv \sigma^* \omega$, so that evaluating A on a given vector at a given point of U would give an antisymmetric matrix. This article uses the convention

$$A \equiv -i\sigma^* \omega \tag{19}$$

instead, so that evaluating A on a given vector at a given point of U gives a symmetric matrix. (In this article, a symmetric matrix is a single real number, because the Lie algebra of $U(1)$ is one-dimensional.) This convention has the benefit of making the results in chapters 17-20 easier to relate to the physics literature.

[21] Article 76708

14 Examples of local potentials

This chapter derives the local potentials defined by the connection one-form from chapter 10 and the two local sections from chapter 11.

First consider the chart and coordinates defined by (12). Given any section σ, the vector fields ∂_x and ∂_y on M define pushforwards $\partial_{x,\sigma}$ and $\partial_{y,\sigma}$ on $\sigma(M) \subset E$ through these conditions:

$$\begin{aligned} \partial_{x,\sigma} f(x_1, ..., x_4) &\equiv \partial_x f\big(\sigma_1(x,y), ..., \sigma_4(x,y)\big) \\ \partial_{y,\sigma} f(x_1, ..., x_4) &\equiv \partial_y f\big(\sigma_1(x,y), ..., \sigma_4(x,y)\big) \end{aligned} \tag{20}$$

for all smooth real-valued functions f defined on points of the section. This gives

$$\partial_{x,\sigma} = \sum_k (\partial_x \sigma_k)\partial_k \qquad\qquad \partial_{y,\sigma} = \sum_k (\partial_y \sigma_k)\partial_k, \tag{21}$$

so the local potential (19) defined using the connection one-form (11) is given by

$$\begin{aligned} iA(\partial_x) \equiv \omega(\partial_{x,\sigma}) &= (\partial_x\sigma_1)\sigma_2 - (\partial_x\sigma_2)\sigma_1 + (\partial_x\sigma_3)\sigma_4 - (\partial_x\sigma_4)\sigma_3 \\ iA(\partial_y) \equiv \omega(\partial_{y,\sigma}) &= (\partial_y\sigma_1)\sigma_2 - (\partial_y\sigma_2)\sigma_1 + (\partial_y\sigma_3)\sigma_4 - (\partial_y\sigma_4)\sigma_3. \end{aligned} \tag{22}$$

Using the local section (13) on the right-hand side gives

$$A(\partial_x) = \frac{-y}{1+\rho^2} \qquad\qquad A(\partial_y) = \frac{x}{1+\rho^2}, \tag{23}$$

so the local potential is

$$A = A(\partial_x)dx + A(\partial_y)dy = \frac{x\,dy - y\,dx}{1+\rho^2}. \tag{24}$$

Now consider the chart and coordinates defined by (14). For the local section defined by (15), following the same pattern as before leads to the local potential

$$\tilde{A} \equiv -i\tilde{\sigma}^*\omega = \frac{\tilde{x}\,d\tilde{y} - \tilde{y}\,d\tilde{x}}{1+\tilde{\rho}^2}. \tag{25}$$

15 Relationship between the local potentials

The intersection between the chart with $w \neq 0$ and the chart with $z \neq 0$ consists of all points for which z and w are both nonzero. In the intersection, the two coordinate systems for M defined in chapter 11 are related to each other by equations (16). Use that relationship in (25) to get

$$\tilde{A} = -\frac{x\,dy - y\,dx}{(1+\rho^2)\rho^2} = \frac{-A}{\rho^2}. \tag{26}$$

Now $\tilde{A}$ is expressed using the same coordinate system as A, we can consider their difference:

$$A - \tilde{A} = \frac{x\,dy - y\,dx}{1+\rho^2}\left(1 + \frac{1}{\rho^2}\right) = \frac{x\,dy - y\,dx}{\rho^2}. \tag{27}$$

Now define ϕ by $x + iy = \rho e^{i\phi}$, as in chapter 12. Use $y/x = \tan\phi$ to get

$$\frac{x\,dy - y\,dx}{x^2} = (1 + \tan^2\phi)d\phi,$$

which may be re-arranged to get

$$\frac{x\,dy - y\,dx}{\rho^2} = d\phi. \tag{28}$$

Use this in (27) to get

$$\tilde{A} = A - d\phi. \tag{29}$$

16 Curvature of the connection

The curvature of a connection one-form ω is given by[22]

$$\Omega(X, Y) = (d\omega)(X_H, Y_H),$$

where X_H, Y_H are the horizontal components of the vector fields X, Y on S^3. When the structure group is abelian, as it is in this article, the exterior derivative $d\omega$ of ω is automatically independent of the vertical components of the vectors X, Y that we feed into it,[22] so in this case we can write

$$\Omega(X, Y) = (d\omega)(X, Y). \tag{30}$$

The exterior derivative of the connection one-form defined by equation (11) is

$$d\omega = 2i(dx_1 \wedge dx_2 + dx_3 \wedge dx_4).$$

More explicitly,

$$(d\omega)(X, Y) \propto dx_1(X)dx_2(Y) - dx_1(Y)dx_2(X) + dx_3(X)dx_4(Y) - dx_3(Y)dx_4(X). \tag{31}$$

To confirm that this is independent of the vertical components of X and Y, set $X = X_V$, where $X_V \equiv L_{12} + L_{34}$ is the vertical vector field defined in chapter 4. Use the identities

$$dx_1(X_V) = -x_2 \qquad dx_2(X_V) = x_1 \qquad dx_3(X_V) = -x_4 \qquad dx_4(X_V) = x_3$$

in equation (31) to get

$$\Omega(X_V, Y) \propto x_1\, dx_1(Y) + x_2\, dx_2(Y) + x_3\, dx_3(Y) + x_4\, dx_4(Y).$$

Equation (10) says that this is zero for all vector fields Y tangent to the total space S^3. This shows that $(d\omega)(X, Y)$ is independent of the vertical components of X, Y, as claimed, with the understanding that we should only consider vector fields X, Y that are tangent to S^3.[23]

[22] Article 76708

[23] Recall the warning at the end of chapter 4.

17 Field strength

The field strength F is the pullback of the curvature two-form Ω by a local section σ,[24,25] defined here with a factor of i as in (19):

$$F \equiv -i\sigma^*\Omega. \tag{32}$$

When the structure group is abelian, the field strength may be written[24]

$$F = dA$$

where A is the local potential (the pullback of the connection ω by the given local section). The field strength of the local potential (24) is

$$\begin{aligned} F &= 2\frac{dx \wedge dy}{1+\rho^2} - d\rho \wedge \frac{x\,dy - y\,dx}{(1+\rho^2)^2} \\ &= 2\frac{dx \wedge dy}{1+\rho^2} - 2(x\,dx + y\,dy) \wedge \frac{x\,dy - y\,dx}{(1+\rho^2)^2} \\ &= 2\frac{dx \wedge dy}{1+\rho^2} - 2\rho^2\frac{dx \wedge dy}{(1+\rho^2)^2} \\ &= \frac{2\,dx \wedge dy}{(1+\rho^2)^2}. \end{aligned} \tag{33}$$

Equation (29) says that this is the same as the field strength of the local potential (25) where the two charts overlap, because $d(d\phi) = 0$.[26] This is consistent with a general result deduced in article 76708: when the structure group is abelian, the field strength is independent of which section is used. As a result, when the structure group is abelian, the field strength is defined on the whole base space, not just on individual charts.

[24] Article 76708

[25] The curvature is a two-form on the total space S^3. The field strength is a two-form on the base space S^2.

[26] The function ϕ itself is defined only modulo 2π, but the identity (28) may be used to check that $d(d\phi) = 0$.

18 Total flux, first method

The field strength F depends on the connection ω, but its integral over the base space – the total flux $\int_M F$ – does not: the total flux is a property of the fiber bundle itself.[27] Changing the connection changes the way the flux is distributed over the base space, but it doesn't change the total.

This chapter calculates the integral of the field strength (33) over the whole base space $M = S^2$. The result is[28]

$$\int_M F = 2\pi. \tag{34}$$

A trivial bundle admits a connection for which $d\omega = 0$, so the total flux (which is independent of the connection) for a trivial bundle must be zero. The nonzero result (34) implies that the Hopf bundle cannot be trivial.

Here's the calculation that gives the result (34):

$$\begin{aligned}
\int_M F &= 2\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; \frac{1}{(1+\rho^2)^2} \\
&= 2\int_0^{2\pi} d\phi \int_0^{\infty} \rho \, d\rho \; \frac{1}{(1+\rho^2)^2} && (x + iy = \rho e^{i\phi}) \\
&= 4\int_0^{\infty} \rho \, d\rho \; \frac{1}{(1+\rho^2)^2} \\
&= 2\pi \int_1^{\infty} ds \; \frac{1}{s^2} && (s \equiv 1 + \rho^2) \\
&= 2\pi
\end{aligned}$$

[27] Collinucci and Wijns (2006), equation (122) and the text below equation (118)

[28] The expression (33) for the field strength is derived from a local potential that is defined only in one chart, but that chart covers almost all of the base space M, excluding only a set of measure zero, so we can still use this to calculate the integral over all of M.

19 Total flux, second method

The result derive in chapter 18 can also be derived another way, using the relationship[29]

$$\int_\Sigma F = \int_\Sigma dA = \int_{\partial\Sigma} A$$

for any two-dimensional part Σ of the chart on which the local potential A is defined, with boundary $\partial\Sigma$.[30] To use this approach, write M as the union of two surfaces Σ and $\tilde{\Sigma}$, where Σ is the part of baseman defined by $|z/w| \leq 1$ and $\tilde{\Sigma}$ is the part defined by $|w/z| \leq 1$. These two surfaces share the same boundary, which is the circle $|z/w| = 1$, so

$$\begin{aligned}\int_M F &= \int_\Sigma F + \int_{\tilde{\Sigma}} F \\ &= \int_\Sigma dA + \int_{\tilde{\Sigma}} d\tilde{A} \\ &= \int_{\partial\Sigma} A + \int_{\partial\tilde{\Sigma}} \tilde{A}\end{aligned} \tag{35}$$

where A and $\tilde{A}$ are the local potentials given in equations (24) and (25), respectively. On the shared boundary, the condition $|z/w| = 1$ allows us to write the coordinates defined in equations (12) and (14) in terms of angular variables ϕ and $\tilde{\phi}$ defined by $x + iy = e^{i\phi}$ and $\tilde{x} + i\tilde{y} = e^{i\tilde{\phi}}$. This gives

$$\int_{\partial\Sigma} A = \frac{1}{2}\int_0^{2\pi} d\phi = \pi \qquad\qquad \int_{\partial\tilde{\Sigma}} \tilde{A} = \frac{1}{2}\int_0^{2\pi} d\tilde{\phi} = \pi,$$

and using these in (35) reproduces the result (34).

[29] This is a special case of **Stokes' theorem** (article 09894).

[30] I'm being cavalier about orientations here, but orientations are important: the direction in which we integrate around the boundary must be consistent with the orientation of the surface integral.

20 Quantization of the total flux

Consider any principal $U(1)$-bundle with base space $M = S^2$. (The Hopf bundle is one example.) This chapter shows that for any such bundle, the total flux must be an integer multiple of 2π, as illustrated by the result (34).

Divide the base space $M = S^2$ into two contractible surfaces U_1 and U_2, whose shared boundary is a closed curve γ. Let A_1 and A_2 be local potentials for these two surfaces, so that

$$\text{total flux} = \int_\gamma A_1 - \int_\gamma A_2. \tag{36}$$

The negative sign in the second term accounts for the fact that we must integrate around the curve γ in opposite directions in the two terms so that their sum gives the total flux, oriented consistently over all of the base space. The local potentials A_1 and A_2 correspond to the same connection but to different local sections (one defined over U_1 and the other over U_2), so they are related to each other by a gauge transformation – a $U(1)$-valued function g of the overlap $U_1 \cap U_2 = \gamma$. Parameterize the curve γ by a real variable t. Since $\gamma(t)$ is a closed curve, the function $g(t)$ must have the form $e^{i\phi(t)}$ for some smooth real-valued function $\phi(t)$ whose initial and final values differ by $2\pi n$ for some integer n, so that the initial and final values of $g(t)$ are equal to each other. Using the result derived in article 76708 for the effect of a gauge transformation on a local potential (and accounting for the factor of $-i$ in the definition (19)), the relationship between A_1 and A_2 is

$$A_1 = A_2 + d\phi.$$

Use this in equation (36) to get

$$\text{total flux} = \int_\gamma (A_2 + d\phi) - \int_\gamma A_2 = \int_\gamma d\phi = 2\pi n, \tag{37}$$

which shows that the total flux must be an integer multiple of 2π.

21 Connection-independence of the total flux

The derivation in chapter 20 showed that the total flux must be an integer multiple of 2π, but it also showed something else: it showed that the total flux is independent of the connection, because the local potential cancels in equation (37). The integer n on the right-hand side of equation (37) must therefore be an intrinsic property of the principal bundle itself, regardless of which connection we choose.

The quantity $-F/2\pi$, whose integral over the base space equals $-n$, is called the **first Chern class**. More precisely and more generally, given a principal G-bundle and an associated vector bundle that uses a linear representation ρ of G,[31] the first Chern class is the quantity $-\text{trace}\big(\rho(F)\big)/2\pi$.[32,33] If $G = U(1)$ and if ρ is faithful, then we can write it more simply as $-F/2\pi$.

Chapter 20 showed that for a $U(1)$ bundle over S^2, the integral of the first Chern class is an integer. That's a special case of a general theorem saying that the **top Chern class** (which involves k factors of the two-form F if the base space is $2k$-dimensional) gives an integer called the **Euler characteristic** or **Euler number** when integrated over the base space.[34,35]

Up to equivalence, one principal $U(1)$-bundle over S^2 exists with Euler number n, for each integer n.[36] Chapters 22-25 will describe the $n \neq 0$ cases in detail.

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[31] Article 70621

[32] Nakahara (1990), section 11.2.1

[33] If the factor of $-i$ were omitted in equations (19) and (32), as in Nakahara (1990) equations (10.6) and (10.37), then the standard coefficient of the first Chern class would be $i/2\pi$, as in Nakahara (1990) equation (11.31b), instead of $-1/2\pi$.

[34] Bott and Tu (1982), theorem 11.6, proposition 11.24, and statement (20.10.6)

[35] Some geometric intuition about the Euler number is given in Wendl (2019), section 1.1; and in Weiss (2018), beginning of section 17.7

[36] Turaev (1992), pages 50-51 (text following corollary 3.3, and at the beginning of part d of section 3)

22 Generalization to any $n \neq 0$: construction

This chapter explains how to construct the nontrivial $U(1)$-bundles over S^2 that were mentioned in chapters 2 and 21, one for each nonzero value of the integer in equation (37).

As in chapter 1, use a pair of complex variables z, w subject to the condition (1) to describe a three-dimensional manifold S^3 embedded in $\mathbb{R}^4$. Let $[z, w]_M$ denote the equivalence class defined by the equivalence relation

$$(\lambda z, \lambda w) \sim (z, w) \qquad \text{for all } \lambda \in U(1), \tag{38}$$

and let $[z, w]_E$ denote the equivalence class defined by the equivalence relation

$$(e^{2\pi i k/n} z, e^{2\pi i k/n} w) \sim (z, w) \qquad \text{for all integers } k, \tag{39}$$

where n is a fixed nonzero integer. As before, take the base space M to be S^3 modulo the equivalence relation (38), so $M = S^2$, but now take the total space E to be S^3 modulo the equivalence relation (39). Then E is still a three-dimensional manifold, but it is distinct from S^3 unless $n = \pm 1$.[37] As before, define the the bundle projection $p : E \to M$ by

$$p(z, w) = [z, w]_M. \tag{40}$$

So far, this defines a fiber bundle whose fiber is diffeomorphic to $U(1)$. To promote it to a principal $U(1)$-bundle, we also need to specify the action of the group $U(1)$ on the total space E. The action is[38]

$$[z, w]_E \times \lambda = [\lambda^{1/n} z, \lambda^{1/n} w]_E. \tag{41}$$

Equation (41) is where the sign of n becomes important. When n is negative, equation (41) may be rewritten using the identity $\lambda^{1/n} = \bar{\lambda}^{1/|n|}$, where $\bar{\lambda}$ is the complex conjugate of λ.

[37] For $|n| \geq 2$, the resulting manifold E is called a **lens space** (Albers *et al* (2019), beginning of section 3).

[38] When $n > 0$, λ has n different nth roots that differ from each other by factors of $e^{2\pi i/n}$, but equation (41) is still unambiguous thanks to the equivalence relation (39).

23 Generalization to any $n \neq 0$: local trivializations

The goal is to show that the principal $U(1)$-bundle constructed in chapter 22 has total flux $2\pi n$. As a step toward that goal, this chapter constructs local trivializations.

Let U_1 be the chart that includes all of the base space except the single point $[0, w]_M$, and let U_2 be the chart that includes all of the base space except the single point $[z, 0]_M$. Just like before, we can use the real and imaginary parts of the ratio z/w as coordinates for U_1, and we can use the real and imaginary parts of the ratio w/z as coordinates for U_2. This works because factors $e^{2\pi ik/n}$ on the left-hand side of (39) cancel each other in these ratios.

Using the coordinate systems defined above to label points of U_1 and U_2, the map

$$[z, w]_E \mapsto \left(z/w,\, w^n/|w|^n\right) \in U_1 \times U(1) \tag{42}$$

is a local trivialization for the part of the bundle over U_1, and the map

$$[z, w]_E \mapsto \left(w/z,\, z^n/|z|^n\right) \in U_2 \times U(1) \tag{43}$$

is a local trivialization for the part of the bundle over U_2. The n-dependence of these maps ensures that they respect the equivalence relation (39) and that they are $U(1)$-equivariant.[39] When n is negative, we can rewrite the map (42) using the identity $(w/|w|)^n = (|w|/\bar{w})^{|n|}$, and similarly for (43). Local trivializations must be invertible, and the inverses of the maps (42)-(43) are[40]

$$\tau_1(u, f) = \left[uf^{1/n}/c,\, f^{1/n}/c\right]_E \qquad c \equiv \sqrt{|u|^2 + |f^{1/n}|^2} \tag{44}$$

and

$$\tau_2(u, f) = \left[f^{1/n}/c,\, uf^{1/n}/c\right]_E \qquad c \equiv \sqrt{|u|^2 + |f^{1/n}|^2}, \tag{45}$$

respectively.

[39] Article 70621

[40] The functions (44)-(45) are unambiguous, because for $n > 0$, the various nth roots of $f \in U(1)$ are related to each other by factors of $e^{2\pi i/n}$, and the equivalence class $[\cdot, \cdot]_E$ is immune to these factors.

24 Generalization to any $n \neq 0$: transition function

Chapter 23 constructed two local trivializations, one over the chart U_1 and one over the chart U_2. Where the charts overlap, the local trivializations are related to each other by a **transition function**.[41,42] The transition function T is defined by[41]

$$\tau_1(u, f) = \tau_2\big(u, T(u)f\big) \tag{46}$$

for all $u \in U_1 \cap U_2$ and all $f \in U(1)$.

Equations (42)-(45) use different coordinate systems to represent points of U_1 and U_2. That's convenient for showing that the inverse maps (44)-(45) exist, but it's inconvenient for solving equation (46). To facilitate solving equation (46), write the local trivializations (42)-(43) like this instead:

$$\tau_1^{-1} : [z, w]_E \mapsto \big([z, w]_M,\ w^n/|w|^n\big) \tag{47}$$
$$\tau_2^{-1} : [z, w]_E \mapsto \big([z, w]_M,\ z^n/|z|^n\big). \tag{48}$$

Use these to write (46) as

$$\tau_1\big([z, w]_M, w^n/|w|^n\big) = \tau_2\big([z, w]_M, Tw^n/|w|^n\big).$$

Compare this to the identity

$$\tau_1\big([z, w]_M, w^n/|w|^n\big) = [z, w]_E = \tau_2\big([z, w]_M, z^n/|z|^n\big)$$

to deduce

$$T\big([z, w]_M\big) = \frac{(z/w)^n}{|z/w|^n}. \tag{49}$$

[41] Article 70621

[42] Some authors call it a **clutching function** or **clutching map**, including Husemoller (1966) and Cohen (2023). The construction of a bundle over S^n from two patches is sometimes called a **clutching construction** (Husemoller (1966)).

25 Generalization to any $n \neq 0$: total flux

Now let's calculate the total flux, using the approach described in chapter 20. Let ω be an arbitrary connection, and let σ_1 and σ_2 be local sections over the charts U_1 and U_2, respectively. Chapter 21 already determined that the total flux will turn out to be independent of the connection ω and of the local sections σ_1 and σ_2, so let's choose the local sections to make the calculation easy: take $\sigma_1(u) = 1$ for all $u \in U_1$, and take $\sigma_2(u) = 1$ for all $u \in U_2$.

The corresponding local potentials are $-i\sigma_1^*\omega$ and $-i\sigma_2^*\omega$. Equation (37) expresses the total flux in terms of the gauge transformation that converts one of these local potentials to the other. If we use the local sections specified in the previous paragraph, then this gauge transformation is the same as the transition function (49).[43] The function (49) may also be written

$$T\big([z,w]_M\big) = e^{in\theta} \qquad \text{with } e^{i\theta} \equiv \frac{z/w}{|z/w|}. \tag{50}$$

To calculate the total flux, take the loop γ in equation (37) to be the equator $|z/w| = 1$ of the base space $M = S^2$. Going once around the equator corresponds to going from $\theta = 0$ to $\theta = 2\pi$, so the phase of the transition function goes from $n\theta = 0$ to $n\theta = 2\pi n$. Use this in equation (37) to deduce that the total flux is $2\pi n$. This works for every nonzero integer $n \neq 0$.[44] The case $n = 1$ is the Hopf bundle that was the focus of most of this article.

[43] This is true no matter what local sections we use, because the integral of the local potential around a closed loop in U_1 or U_2 is independent of the local section (chapter 17).

[44] Equation (41) says that changing the sign of n corresponds to reversing the orientation of the action of the group $U(1)$, which changes the sign of the total flux.

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