Cosmology in N-Dimensional Spacetime: the Friedmann Equations

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Abstract The **Friedmann equations** are consequences of the Einstein field equation when the geometry of spacetime and the matter that occupies spacetime are both assumed to be homogeneous and isotropic. When matter is absent, the solutions include **de Sitter spacetime** and **anti de Sitter spacetime** as special cases, depending on the sign of the cosmological constant. This article derives the Friedmann equations in a spacetime with any number of dimensions. The four-dimensional version of this derivation is a standard exercise in the study of general relativity because of its importance in cosmology.

Contents

1 Introduction and conventions

General relativity is about consequences of the Einstein field equation[1]

$$R_{ab} - \frac{1}{2}g_{ab}R = \kappa T_{ab} + g_{ab}\Lambda, \tag{1}$$

where R_{ab} is the Ricci tensor, $R \equiv g^{ab}R_{ab}$ is the scalar curvature, T_{ab} is the **matter tensor** (also called the **stress-energy tensor**), Λ is the cosmological constant, and $\kappa > 0$ is proportional to the gravitational constant.[2] The *Friedmann equations* are derived from (1) by using a simple ansatz for the metric tensor[3] and a simple ansatz for the matter tensor.[4] This article describes each ansatz, derives the Friedmann equations, and explores a few of their consequences.[5]

The number of spacetime dimensions is $D+1$. The coordinates will be denoted $(t, \mathbf{x})$, where t is the "time" coordinate and $\mathbf{x}$ is the list of D "space" coordinates. The notation $\mathbf{x}\cdot\mathbf{y}$ denotes the sum of the products of corresponding coordinates from $\mathbf{x}$ and $\mathbf{y}$. The abbreviations

$$\mathbf{x}^2 \equiv \mathbf{x}\cdot\mathbf{x} \qquad r \equiv \sqrt{\mathbf{x}^2} \qquad dt^2 \equiv (dt)^2 \qquad d\mathbf{x}^2 \equiv (d\mathbf{x})^2$$

will also be used, where dt and $d\mathbf{x}$ are coordinate differentials.

A spacetime index will be written using a letter from the beginning of the alphabet, like a, b, c, d. Sometimes symbols like $\bullet$ and $\times$ will be used as spacetime indices to help make equations with multiple indices easier to read quickly. The letter j or k denotes an index corresponding to a "space" coordinate, and 0 is the index-value corresponding to the "time" coordinate t.

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[1] This article uses the mostly minus convention for the metric tensor (article 48968). The sign conventions used for other quantities are standard (article 80838).

[2] Article 99922 shows that κ must be positive in order for gravity to be attractive.

[3] Section 2

[4] Section 8

[5] Cosmology is about much more than just the Friedmann equations. Reviews of cosmology include Workman *et al* (2023), Green *et al* (2022), Cline (2018), Baumann (2018), Carroll (2000).

2 The ansatz for the spacetime metric

The Friedmann equations are derived from general relativity, using an ansatz for the metric tensor called a **Friedmann-Lemaître-Robertson-Walker** model, which will be abbreviated **FLRW** in this article.[6] Using the notation summarized in section 1, the FLRW metric is defined implicitly by this expression for the proper time increment $d\tau$ along any timelike worldline:[7]

$$\boxed{d\tau^2 = dt^2 - a^2(t)ds_D^2} \tag{2}$$

where ds_D is the line element of a D-dimensional[8] **maximally symmetric** space with euclidean signature. *Maximally symmetric* means homogeneous (the same at all points) and isotropic (the same in all directions).[9] We can take the line element ds_D to be

$$ds_D^2 = b^2(r)\, d\mathbf{x}^2 \tag{3}$$

with

$$b(r) \equiv \frac{2}{1+\epsilon r^2} \qquad\qquad \epsilon = \pm 1 \text{ or } 0. \tag{4}$$

The independent coordinates are t and $\mathbf{x}$. The letter r is an abbreviation for $\sqrt{\mathbf{x}^2}$. The t-dependence of the **(cosmic) scale factor** $a(t)$ will be determined by solving the Einstein field equation.

The geometry defined by (3)-(4) has manifest rotation symmetry about $r = 0$. The point $r = 0$ appears to be special, but that's an artifact of the coordinate system. Article 96560 shows that the geometry defined by (3)-(4) is maximally symmetric, not merely rotation-symmetric about one point.

[6] The names **FRW** (`https://ncatlab.org/nlab/show/FRW+model`) and **Robertson-Walker** (Wald (1984), Blau (2022)) are also common.

[7] In a superscript or subscript, the letter a or b denotes an index (section 1). Otherwise, a and b denote the functions appearing in equations (2)-(3).

[8] This is the reason for the subscript D on ds_D.

[9] Article 96560

3 Another common way to write the spatial part

Define

$$\mathbf{u} \equiv \frac{\mathbf{x}}{r} \tag{5}$$

so that

$$d\mathbf{x}^2 = dr^2 + r^2\, d\mathbf{u}^2.$$

We could express $d\mathbf{u}^2$ in terms of $D-1$ angular coordinates,[10] but that won't be necessary here. Use this expression for $d\mathbf{x}^2$ to get

$$\big(b(r)\, d\mathbf{x}\big)^2 = b^2(r)\, dr^2 + \rho^2\, d\mathbf{u}^2 \tag{6}$$

with $\rho \equiv b(r)r$. The definition of ρ implies

$$d\rho = \frac{1-\epsilon r^2}{1+\epsilon r^2} b\, dr \qquad\qquad 1-\epsilon\rho^2 = \left(\frac{1-\epsilon r^2}{1+\epsilon r^2}\right)^2.$$

Use these in the right-hand side of (6) to get[11]

$$\big(b(r)\, d\mathbf{x}\big)^2 = \frac{d\rho^2}{1-\epsilon\rho^2} + \rho^2\, d\mathbf{u}^2.$$

This is another common way of writing the spatial part of the line element in the literature about cosmology. This article will use the form that was described in section 2.

[10] For $D = 3$, we could use $d\mathbf{u}^2 = d\theta^2 + \sin^2\theta\, d\phi^2$.

[11] This is a special case of a result derived in article 96560 for a more general family of maximally symmetric spaces.

4 The ambiguous distinction between time and space

The same spacetime geometry may be described in different ways, using different coordinate systems. The geometry of spacetime determines which worldlines are timelike and which ones are spacelike, independently of what coordinate system we use.[12] If we use a coordinate system $(t, \mathbf{x})$ in which the partial derivative with respect to t is a timelike vector field,[13] then we often refer to the coordinate t as "time," and we often refer to a constant-t hypersurface as "space." Langauge like this can be convenient, and this article will use it, but we should remember that this distinction between "time" and "space" is an artifact of the coordinate system, not an intrinsic property of the geometry of spacetime.

Example: the geometry of **de Sitter spacetime** can be described using a line element of the form (2) with $\epsilon = 1$, or with $\epsilon = 0$, or with $\epsilon = -1$, depending on what coordinate system we use. Section 17 will give more information about this non-obvious fact.

[12] Article 48968

[13] Article 09894 introduces the relationship between partial derivatives and vector fields.

5 Cosmological redshift

This section highlights a property of the ansatz (2)-(3).

Consider two objects with worldlines of the form

$$\mathbf{x} = \text{constant}. \tag{7}$$

Thanks to homogeneity and isotropy, we might as well take one of the objects to be at the origin

$$\mathbf{x} = \mathbf{0},$$

and we might as well assume that the other object's $\mathbf{x}$ has only one non-zero component, hereafter denoted x_R. Equation (2) shows that each of these worldlines is timelike and that its proper time is the same as the coordinate time t. Section 7 will show that these worldlines are geodesics, so these objects are in free-fall.

Now suppose that the object at the origin transmits two consecutive light pulses to the object at x_R. Let $\delta\tau_T$ be the proper time interval between transmissions according to the transmitting object. The goal is to compute $\delta\tau_R$, the proper time interval between receptions according to the receiving object.

For a null geodesic ($d\tau = 0$), equation (2) gives

$$\frac{dt}{a(t)} = \pm b(x)\, dx. \tag{8}$$

The sign describes the direction of motion of the light pulse. For the rest of this section, take the sign to be positive. Use this notation:

- Pulse #1 is transmitted at coordinate-time t_T.
- Pulse #1 is received at coordinate-time t_R.
- Pulse #2 is transmitted at coordinate-time $t_T + \delta t_T$.
- Pulse #2 is received at coordinate-time $t_R + \delta t_R$.

Equation (8) gives

$$\int_{t_T}^{t_R} \frac{dt}{a(t)} = \int_0^{x_R} dx\ b(x) \qquad\qquad \int_{t_T+\delta t_T}^{t_R+\delta t_R} \frac{dt}{a(t)} = \int_0^{x_R} dx\ b(x).$$

The two left-hand sides are equal to each other. Equate them and then add $\int_{t_R}^{t_T+\delta t_T} \frac{dt}{a(t)}$ to both sides to get

$$\int_{t_T}^{t_T+\delta t_T} \frac{dt}{a(t)} = \int_{t_R}^{t_R+\delta t_R} \frac{dt}{a(t)}.$$

For sufficiently small δt_T and δt_R, this implies

$$\frac{\delta t_T}{a(t_T)} \approx \frac{\delta t_R}{a(t_R)}.$$

According to equation (2), the coordinate time t coincides with the proper time for any object with worldline of the form (7), so the ratio of the proper intervals is

$$\boxed{\frac{\delta\tau_T}{\delta\tau_R} = \frac{\delta t_T}{\delta t_R} \approx \frac{a(t_T)}{a(t_R)}.} \tag{9}$$

This implicitly depends on the spatial displacement x_R, because the time-difference $t_R - t_T$ does.

Equation (9) describes an effect called **cosmological redshift**. If the unverse is expanding, so that $a(t_R) > a(t_T)$, then equation (9) says that the frequency of the light received by the object at $x = x_R$ is lower than the frequency of the light that was emitted by the object at $x = 0$.

6 A more general coordinate system

Section 7 will calculate the components of the Ricci tensor R_{ab} for the metric that was defined in section 2. The calculation will be done using a more general coordinate system than the one that was used in that section. In the more general coordinate system, equation (3) becomes

$$ds_D^2 = h_{jk}(\mathbf{x})dx^j\,dx^k \qquad\qquad h_{jk} = h_{kj}, \tag{10}$$

so the full FLRW metric (2) becomes

$$d\tau^2 = dt^2 - a^2(t)h_{jk}(\mathbf{x})dx^j\,dx^k. \tag{11}$$

As in (3), the spatial coordinates are denoted $\mathbf{x} = (x^1, x^2, ..., x^D)$, but now $h_{jk}(\mathbf{x})$ are the components of the spatial metric (which is still maximally symmetric) in an arbitrary coordinate system (which may obscure the maximal symmetry). Each index j and k takes values in $\{1, 2, ..., D\}$, and sums over the repeated indices j, k are implied. After calculating the Ricci tensor, the result will be specialized back to the coordinate system that was used in section 2.

Arranging the calculation this way allows us to use a general result for the Ricci tensor of any maximally symmetric space. Article 96560 highlights that general result and checks it for the geometry defined by (3)-(4).[14]

[14] Article 96560 does this by specializing and even more general result that applies to any **conformally flat** metric. The metric defined by (3) is manifestly conformally flat.

7 The geodesic equations and the Ricci tensor

This section derives the geodesic equations using a lagrangian method introduced in article 33547, extracts the connection coefficients from these geodesic equations, and uses these connection coefficients to calculate the components of the Ricci tensor. In this section, a dot over a function's name (a letter) denotes a derivative of that function with respect to its argument. The scale factor a is a function of the t, so $\dot{a} \equiv da/dt$. In the geodesic equations, the coordinates t and x^k are functions of the worldline parameter λ, so $\dot{t}$ and $\dot{x}^k$ mean $dt/d\lambda$ and $dx^k/d\lambda$.

The geodesic equations are

$$\frac{d}{d\lambda}\frac{\delta L}{\delta \dot{t}} = \frac{\delta L}{\delta t} \qquad\qquad \frac{d}{d\lambda}\frac{\delta L}{\delta \dot{x}^k} = \frac{\delta L}{\delta x^k}, \tag{12}$$

where the lagrangian corresponding to (11) is

$$L = \dot{t}^2 - a^2(t)h_{jk}(\mathbf{x})\dot{x}^j\,\dot{x}^k. \tag{13}$$

The connection coefficients Γ^a_{bc} may be deduced by comparing these geodesic equations to the general form[15,16]

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0. \tag{14}$$

According to (13), the variational derivatives in equations (12) are

$$\begin{aligned} \frac{\delta L}{\delta \dot{t}} &= 2\dot{t} & \qquad \frac{\delta L}{\delta t} &= -2\dot{a}ah_{jk}\dot{x}^j\,\dot{x}^k \\ \frac{\delta L}{\delta \dot{x}^\ell} &= -2a^2 h_{\ell k}\dot{x}^k & \qquad \frac{\delta L}{\delta x^\ell} &= -a^2\partial_\ell h_{jk}\dot{x}^j\dot{x}^k. \end{aligned}$$

[15] A letter a, b, c denotes a spacetime index. A letter j, k, ℓ denotes a space-only index.

[16] In a superscript or subscript, a or b is an index. Otherwise, a and b are the functions defined in section 2.

Substitute these into (12), evaluate the derivatives $d/d\lambda$ in (12), and re-arrange to get the geodesic equations

$$\boxed{\ddot{t} + \dot{a}ah_{jk}\dot{x}^j\dot{x}^k = 0} \qquad \boxed{\ddot{x}^\ell + \frac{2\dot{a}}{a}\dot{t}\dot{x}^\ell + \hat{\Gamma}^\ell_{jk}\dot{x}^j\dot{x}^k = 0} \tag{15}$$

where $\hat{\Gamma}^\ell_{jk}$ are the connection coefficients for the D-dimensional geometry defined by (10). Equations (15) show that the only non-zero coefficients in (14) are[17]

$$\Gamma^0_{jk} = \dot{a}ah_{jk} \qquad \Gamma^k_{0\ell} = \Gamma^k_{\ell 0} = \frac{\dot{a}}{a}\delta^\ell_k \qquad \Gamma^\ell_{jk} = \hat{\Gamma}^\ell_{jk}. \tag{16}$$

The Ricci tensor is[18,19,20]

$$R_{ab} = \partial_\bullet \Gamma^\bullet_{ab} - \partial_a \Gamma^\bullet_{\bullet b} + \Gamma^\times_{\times\bullet}\Gamma^\bullet_{ab} - \Gamma^\times_{a\bullet}\Gamma^\bullet_{\times b}. \tag{17}$$

Use equations (16) in (17) to get

$$\boxed{R_{00} = \frac{-\ddot{a}}{a}D} \qquad \boxed{R_{0k} = R_{k0} = 0} \tag{18}$$

$$R_{jk} = (D-1)\dot{a}^2 h_{jk} + a\ddot{a}h_{jk} + \hat{R}_{jk}, \tag{19}$$

where $\hat{R}_{jk}$ are the components of the Ricci tensor for the D-dimensional geometry defined by (10). These are given by an expression of the same form as (17), but with $\hat{R}_{jk}$ and $\hat{\Gamma}$ in place of R_{jk} and Γ. Article 96560 derives the result $\hat{R}_{jk} = (D-1)\epsilon h_{jk}$. Use this in (19) to get[21]

$$\boxed{R_{jk} = \big((D-1)(\dot{a}^2 + \epsilon) + a\ddot{a}\big)h_{jk}.} \tag{20}$$

[17] Blau (2022), equation (35.2)

[18] Article 03519

[19] The symbols $\bullet$ and $\times$ are used here as spacetime indices, and sums over these indices are implied.

[20] ∂_a denotes the partial derivative with respect to the ath coordinate.

[21] Blau (2022), equation (35.6)

8 The ansatz for the matter tensor

When the D-dimensional geometry (10) is specialized to (3), the results (18) and (20) say that the Ricci tensor has the form

$$R_{ab} = f(t)u_a u_b + g(t)\Delta_{ab}$$

for functions f, g that depend only on t, with

$$\Delta_{ab} \equiv g_{ab} - u_a u_b$$

and u is this timelike unit vector:[22]

$$u_a = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

The Einstein field equation (1) says that the matter tensor T_{ab} must also have that form, but with different t-dependent coefficients:[23]

$$T_{ab} = \rho(t)u_a u_b - p(t)\Delta_{ab}. \tag{22}$$

The coefficient $\rho = T_{00}$ is the energy density,[24] and section 9 will show that the other coefficient p may be interpreted as pressure. A stress-energy tensor of the form (22) is said to describe a **perfect fluid**.[25]

Section 11 will derive the Friedmann equations from (1) by using the ansatz (2) for the metric tensor and the ansatz (22) for the components of the matter tensor.

[22] This is different than the u that was defined in (5). That one had D components. This one has $D + 1$.

[23] The ansatz (22) can also be motivated without using the results (18) and (20), just by considering the symmetry of the ansatz (2)-(4) for the metric tensor (Blau (2022), section 35.3).

[24] In the context of a given spacetime metric, the integral of T^{00} over all of space may be interpreted as the system's total energy. This is illustrated in articles 41182 and 78463. With the metric defined by (2), $T^{00} = T_{00} = T^0_0$.

[25] The review Kovtun (2012) puts this in a larger context. Section 1.2 in that review addresses nonrelativistic hydrodynamics, with a comment about the meaning of *perfect fluid*. Section 1.3 in that review addresses relativistic hydrodynamics, including comments about how a general symmetric tensor T_{ab} may be decomposed with respect to a given timelike vector field u and how the terms in that decomposition relate to fluid dynamics.

9 Pressure

One way to motivate interpreting the quantity p in (22) as pressure comes from a conservation law implied by the field equation (1). Write equation (1) as[26]

$$G^{ab} = \kappa T^{ab} + g^{ab}\Lambda \tag{23}$$

with $G^{ab} \equiv R^{ab} - \frac{1}{2}g^{ab}R$. The tensor with components G^{ab} is called the **Einstein tensor**. It satisfies $\nabla_a G^{ab} = 0$ identically, where ∇ is the covariant derivative.[27] One of the defining properties of the covariant derivative is $\nabla_a g_{bc} = 0$, where g_{bc} are the components of the metric tensor, which implies $\nabla_a g^{ab} = 0$. Consistency of these identities with equation (23) then implies the conservation law

$$\nabla_a T^{ab} = 0. \tag{24}$$

This can be used to motivate interpreting the quantity p in (22) as pressure. For that purpose, consider the kth component of (24) for some $k \neq 0$:

$$\nabla_0 T^{0k} + \nabla_j T^{jk} = 0. \tag{25}$$

For any given point x_p in spacetime, we can choose a coordinate system that puts metric tensor g_{ab} into the form[28]

$$g_{ab}(x) = \eta_{ab} + O\big((x - x_p)^2\big),$$

where $\eta_{ab} = (1, -1, -1, ..., -1)$ is the Minkowski metric. In such a coordinate system, equation (25) reduces to

$$\partial_0 T^{0k} + \partial_j T^{jk} = 0 \tag{26}$$

at the point x_p, where ∂_0 and ∂_j are partial derivatives with respect to the "time" and "space" coordinates, respectively. The integral of T^{0k} over space would be the

[26] Indices are raised and lowered using the metric tensor and its inverse. This notation is standard in physics.

[27] Article 03519 introduces the covariant derivative, and article 37501 shows that $\nabla_a G^{ab} = 0$.

[28] Article 48968

kth component of the total momentum,[29] so T^{0k} itself is the kth component of the momentum density. Equation (26) relates the rate of change of the momentum density to the quantity $\partial_j T^{jk}$. If $T^{jk} = -p(x)\eta^{jk}$, like in the ansatz (22) but with a p that may vary in space, then (26) becomes[30]

$$\partial_0 T^{0k} + \partial_k p = 0$$

at the point x_p. This says that the rate of change of the momentum density is equal to the negative gradient of p, which is consistent with interpreting p as pressure: if the pressure is not constant in space, then it exerts a push toward the direction of decreasing pressure. In the ansatz (22), the gradient is zero because the pressure is homogeneous, but the interpretation still applies because zero gradient is a special case of arbitrary gradient.

[29] This is illustrated in articles 41182 and 78463.

[30] This uses the identity $-\eta^{jk}\partial_j = \partial_k$.

10 Special case: dust

When $p = 0$, the ansatz (22) describes a perfect fluid with zero pressure, also called **dust**. This is an important special case, because (on cosmological scales) the pressure p is negligible today, even though it was significant in the early universe. This section gives a little more insight into the special case $p = 0$.

When $T^{ab} = \rho u^a u^b$, equation (24) is equivalent to the pair of equations

$$\nabla_\bullet(\rho u^\bullet) = 0 \qquad\qquad u^\bullet \nabla_\bullet u^b = 0. \tag{27}$$

These two equations clearly imply (24). To prove the converse, start with the identity

$$\nabla_\bullet T^{\bullet b} = u^b \nabla_\bullet(\rho u^\bullet) + \rho u^\bullet \nabla_\bullet u^b. \tag{28}$$

The fact that u is a unit vector implies[31]

$$u_\bullet \nabla_a u^\bullet = 0, \tag{29}$$

so contracting (28) with u_b and using (24) gives the first equation in (27). Using that result in (28) and using (24) again gives the second equation in (27).

The second equation in (27) says that the vector field u is tangent to a congruence of geodesics.[32] The first equation in (27) says that $\rho(x)$ follows these geodesics. Intuitively, if we think of $\rho(x)$ as the density of a fluid made of grains of dust, then equations (27) say that the worldline of each grain of dust is a geodesic. Equation (21) is consistent with this, because (21) describes a timelike geodesic when the geometry of spacetime is defined by (2)-(3).

[31] To deduce this, use $u_\bullet u^\bullet = 1$ to get $0 = \nabla_a(u_\bullet u^\bullet) = (\nabla_a u_\bullet)u^\bullet + u_\bullet \nabla_a u^\bullet$. Then use the fact that the covariant derivative ∇ commutes with the metric tensor to deduce that the two terms on the right-hand side of this equation are equal to each other, so this equation implies that both of them are zero.

[32] Wald (1984), equation (3.3.1)

11 The Friedmann equations

Using equation (22) for the matter tensor and the ansatz (2) for the metric tensor gives the **Friedmann equations**, a pair of differential equations for the time-dependent scale factor $a(t)$ in (2). One of the Friedmann equations comes from the 00 component of (1), and the other one comes from the trace of (1), which is

$$\frac{1-D}{2}R = \kappa g^{ab}T_{ab} + (D+1)\Lambda. \tag{30}$$

The ansatz (2) for the metric and the results (18) and (20) for the Ricci tensor give this result for the scalar curvature:

$$\begin{aligned} R &= R_{00} - \frac{1}{a^2b^2}\sum_j R_{jj} \\ &= -2\frac{\ddot{a}}{a}D - \left(\frac{\epsilon}{a^2} + \left(\frac{\dot{a}}{a}\right)^2\right)(D-1)D, \end{aligned} \tag{31}$$

and that gives

$$R_{00} - \frac{1}{2}g_{00}R = \frac{(D-1)D}{2}\left(\frac{\epsilon}{a^2} + \left(\frac{\dot{a}}{a}\right)^2\right).$$

For the right-hand sides of equations (1) and (30), equations (22), (21), and (2) give

$$\kappa T_{00} + g_{00}\Lambda = \kappa\rho + \Lambda \qquad\qquad \kappa g^{ab}T_{ab} + g^{ab}g_{ab}\Lambda = \kappa(\rho - Dp) + (D+1)\Lambda. \tag{32}$$

Use these in the 00 component of (1) to get the first Friedmann equation

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 + \frac{\epsilon}{a^2} = \frac{2}{(D-1)D}(\kappa\rho + \Lambda).} \tag{33}$$

Using (31) and (32) in equation (30) gives

$$\frac{D-1}{2}\left(2\frac{\ddot{a}}{a}D+\left[\frac{\epsilon}{a^2}+\left(\frac{\dot{a}}{a}\right)^2\right](D-1)D\right)=\kappa\,(\rho-Dp)+(D+1)\Lambda.$$

After using (33) to rewrite the left-hand side, this becomes

$$(D-1)\left(\frac{\ddot{a}}{a}D+\kappa\rho+\Lambda\right)=\kappa\,(\rho-Dp)+(D+1)\Lambda,$$

which may be rearranged to get the second Friedmann equation:[33]

$$\boxed{\frac{\ddot{a}}{a}=\frac{2\Lambda-(D-2)\kappa\rho-D\kappa p}{(D-1)D}.} \tag{34}$$

Subtracting equation (33) from equation (34) gives[34]

$$\frac{d}{dt}\left(\frac{\dot{a}}{a}\right)=\frac{\epsilon}{a^2}-\frac{\kappa\rho+\kappa p}{D-1}. \tag{35}$$

The t-dependent function

$$H\equiv\frac{\dot{a}}{a} \tag{36}$$

is called the **Hubble parameter**.

[33]Equations (33) and (34) are shown for $\Lambda=0$ in Blau (2022), equations (35.111) and (35.113). Setting $D=3$ in equations (33) and (34) reproduces equations (21.8) and (21.9) in Olive (2023).

[34]Equations (33) and (35) match equations (2.5)-(2.6) in Chen, Gibbons, and Li (2014) for all D.

12 A conservation law

The Einstein field equation (1) implies that T_{ab} is conserved.[35] In particular, the Friedmann equations imply a conservation law for ρ and p. To derive it, multiply equation (33) by a^2, take the derivative of the resulting equation with respect to t, and then use equation (34) to eliminate $\ddot{a}$. The terms involving Λ cancel, and what remains can be rearranged to give the conservation law[36]

$$\boxed{\dot{\rho} = -(\rho + p)D\,\frac{\dot{a}}{a}.} \tag{37}$$

This can also be derived using the conservation law $\nabla_a T^{ab} = 0$ by itself, without using the Friedmann equations. Using an arbitrary coordinate system, write

$$T^{ab} = \rho u^a u^b - \Delta^{ab} p \qquad\qquad \Delta^{ab} \equiv g^{ab} - u^a u^b,$$

and consider the contraction of $\nabla_a T^{ab}$ with u^b:

$$\begin{aligned}
u_b \nabla_a T^{ab} &= u_b \nabla_a\big(\rho u^a u^b\big) - u_b p \nabla_a \Delta^{ab} && \text{(because } u_b \Delta^{ab} = 0\text{)}\\
&= u^a \nabla_a \rho + (\rho + p) u_b \nabla_a (u^a u^b) && \text{(because } \nabla_a g^{bc} = 0\text{)}\\
&= u^a \nabla_a \rho + (\rho + p) \nabla_a u^a && \text{(equation (29))}\\
&= u^a \partial_a \rho + (\rho + p)(\partial_a u^a + \Gamma^{\bullet}_{a\bullet} u^a) && \text{(definition of } \nabla_a\text{)}.
\end{aligned}$$

This holds for any metric tensor and any unit vector field u in any coordinate system. Now use the connection coefficients in section 7 and specialize to the coordinate system in which u is given by (21) to get[37]

$$u_b \nabla_a T^{ab} = \partial_0 \rho + (\rho + p) D \frac{\dot{a}}{a}.$$

This shows that the condition $\nabla_a T^{ab} = 0$ implies (37).

[35] Article 37501

[36] Blau (2022), equation (35.114)

[37] Notice that $\nabla_a u^a \neq 0$ even though $\partial_a u^a = 0$.

13 Solutions with dust

When $p = 0$, equation (37) reduces to

$$\frac{\dot{a}}{a} = -D\frac{\dot{\rho}}{\rho},$$

which implies

$$\rho(t) \propto \frac{1}{a^D(t)}. \tag{38}$$

This describes a universe filled with dust particles that remain at fixed values of the spatial coordinates, so the density scales with $a(t)$.

In the more specific case $\epsilon = \Lambda = p = 0$, equations (33)-(34) are satisfied by

$$a(t) = C\,t^{2/D} \qquad\qquad \rho(t) = \frac{\rho_1}{a^D(t)} \tag{39}$$

with

$$C^D = \frac{\kappa\rho_1 D}{2(D-1)}.$$

This universe begins at $t = 0$ and expands forever.

When $\epsilon = 1$ and $\Lambda = p = 0$, equations (33) and (34) have a solution in which a and ρ are both constants – a closed universe that neither expands nor contracts.

14 Dark energy and dark matter

Given the ansatz for T_{ab} in section 8, the right-hand side of the field equation (1) involves three unspecified quantities that don't appear in the metric in section 2: the energy density $\rho(t)$, the pressure $p(t)$, and the cosmological constant Λ. It actually involves only two combinations of those three quantities, because equation (22) gives

$$T_{00} = \kappa\rho + \Lambda \qquad T_{jk} = (\kappa p - \Lambda)a^2 b^2 \delta_{jk} \qquad T_{0k} = T_{k0} = 0,$$

where $a(t)$ and $b(r)$ are the functions from section 2. The two combinations are the **effective energy density** $\rho + \Lambda/\kappa$ and the **effective pressure** $p - \Lambda/\kappa$. In terms of these combinations, the second Friedmann equation (34) is

$$\frac{\ddot{a}}{a} = \frac{-(D-2)(\kappa\rho + \Lambda) - D(\kappa p - \Lambda)}{(D-1)D}. \tag{40}$$

We could absorb Λ into the definitions of ρ and p. For accounting purposes, though, these contributions are often distinguished from each other:

- Pressure due to known types of matter and radiation, which affects only the combination $p - \Lambda/\kappa$.
- Energy (mass) density due to known types of matter and radiation, which affects only the combination $\rho + \Lambda/\kappa$.
- **Dark matter**, defined as any contribution to $\rho + \Lambda/\kappa$ that doesn't significantly affect $p - \Lambda/\kappa$ and whose existence has so far only been inferred through its gravitational effects.
- **Dark energy**, defined as anything that affects both $\rho + \Lambda/\kappa$ and $p - \Lambda/\kappa$ with equal magnitude and opposite signs and that is not included in the other categories, like Λ/κ. Calling it *dark energy* instead of *cosmological constant* allows for the possibility that it might not be constant.[38]

[38] Section 7.2 in Green *et al* (2022); first paragraph in Peebles and Ratra (2002)

15 The value of Λ

Equation (40) shows that a positive effective energy density and a positive effective pressure both tend to decrease the value of $\dot{a}$ over time. If the universe is expanding ($\dot{a} > 0$), then they both tend to make the rate of expansion slow down over time ($\ddot{a} < 0$).

We have known for a long time that the real universe is expanding ($\dot{a} > 0$),[39] and we more recently accumulated compelling evidence that the rate of expansion is currently increasing ($\ddot{a} > 0$).[40] According to equation (40), this implies

$$(D-2)(\kappa\rho + \Lambda) + D(\kappa p - \Lambda) < 0.$$

The pressure p due to matter and radiation is currently negligible on cosmological scales, so Λ must be positive. Observations are consistent with[41,42]

$$\epsilon = 0 \qquad \frac{\Lambda}{\kappa\rho + \Lambda} \approx 0.7 \qquad \frac{\kappa\rho}{\kappa\rho + \Lambda} \approx 0.3,$$

so Λ is larger than $\kappa\rho$.

Results like this are more often expressed as $\Omega_\Lambda \approx 0.7$ and $\Omega_m \approx 0.3$ with

$$\Omega_\Lambda \equiv \frac{\Lambda/\kappa}{\rho_c} \qquad \Omega_m \equiv \frac{\rho}{\rho_c},$$

where the **critical density** ρ_c is the value of ρ that would satisfy the Friedman equation (33) if $\epsilon = \Lambda = 0$. Explicitly:

$$\rho_c \equiv \frac{(D-1)D}{2\kappa}\left(\frac{\dot{a}}{a}\right)^2.$$

[39] A small but significant discrepancy exists between the rate of expansion inferred from different kinds of measurements (Perivolaropoulos and Skara (2021), figure 12).

[40] Example: Kowalski *et al* (2008), figure 15. Reviews include Frieman *et al* (2008) and section 3 in Carroll (2001). Krauss *et al* (2007) focuses on uncertainties in some of the evidence and cites other reviews of the evidence.

[41] Lahav and Liddle (2023), table 25.1

[42] The density ρ is understood to include both ordinary matter and dark matter (section 14). Observations indicate that it is *mostly* dark matter (Lahav and Liddle (2023), section 25.4, first paragraph, with the notation defined in section 25.1.1).

16 Vacuum solutions

When $\rho = p = 0$ ($T_{ab} = 0$), the Friedmann equations (33) and (34) reduce to

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\epsilon}{a^2} = \frac{2\Lambda}{(D-1)D} \tag{41}$$

$$\frac{\ddot{a}}{a} = \frac{2\Lambda}{(D-1)D}. \tag{42}$$

For any positive real number $s > 0$, these equations are invariant under

$$a \to sa \qquad\qquad t \to st \qquad\qquad \Lambda \to \Lambda/s^2,$$

so we could take $\Lambda \in \{\pm 1, 0\}$ without loss of generality. Use the abbreviation

$$\lambda \equiv \left(\frac{2}{(D-1)D}|\Lambda|\right)^{1/2} \geq 0. \tag{43}$$

Cases:

- If $\Lambda = 0$, then equation (41) implies $\epsilon \leq 0$. The case $\epsilon = 0$ gives $\dot{a} = 0$, which is flat spacetime. If $\epsilon < 0$, then $\dot{a}$ is a nonzero constant.

- If $\Lambda > 0$, then equation (42) says that a is a linear combination of $e^{\lambda t}$ and $e^{-\lambda t}$, so $a = c_1 e^{\lambda t} + c_2 e^{-\lambda t}$. Then equation (41) implies $\epsilon = 4\lambda^2 c_1 c_2$, so ϵ can be positive, negative, or zero. All three of these cases give **de Sitter (dS) spacetime**, written in different coordinate systems.[43]

- If $\Lambda < 0$, then equation (42) says that $a = c_1 \cos(\lambda t + \phi)$ for real constants c_1 and ϕ. Then equation (41) implies $\epsilon = -\lambda^2 c_1^2 < 0$. This is **anti de Sitter (AdS) spacetime**.[44]

[43] Section 17

[44] Blau (2022), sections 37.5 and 39.3.4

17 de Sitter spacetime

Section 16 mentioned that when $\Lambda > 0$ and $T_{ab} = 0$, solutions with $\epsilon \in \{+1, 0, -1\}$ all describe the same spacetime geometry (de Sitter spacetime).[45] They look different because they use different coordinate systems and because they don't all cover the whole spacetime. The $\epsilon = 1$ version covers the whole spacetime except for a set of measure zero,[46] and the $\epsilon = 0$ and $\epsilon = -1$ versions only cover smaller parts.[47]

For any D, de Sitter spacetime may be defined as a special $D + 1$-dimensional submanifold embedded in an ambient $D + 2$-dimensional Minkowski spacetime, from which it inherits its intrinsic geometry.[48] For the case $D = 1$ (two-dimensional de Sitter spacetime), this embedding is illustrated graphically in Moschella (2005), showing which parts of de Sitter spacetime are covered by each case $\epsilon \in \{+1, 0, -1\}$.

The three options $\epsilon \in \{1, 0, -1\}$ use different "time" coordinates (denoted t in equation (2)), giving different definitions of "space" (hypersurfaces on which t is constant). The case $\epsilon = 0$ is called the **planar coordinate system**,[49] because in this case "space" is flat. It's also called the **inflationary coordinate system**,[50] because the scale factor $a(t)$ in (2) grows exponentially in this case.[51] This coordinate system covers only part of de Sitter spacetime.

This is a solution of the field equations with $\Lambda > 0$ when $T_{ab} = 0$, or an approximate solution when κT_{ab} is negligible compared to Λ. That condition is not satisfied today, but it might be eventually if the scale factor grows enough to make $\kappa\rho$ negligible compared to Λ (equation (38)). The idea that a future part of the real universe might be approximated by a corresponding part of de Sitter spacetime is prominent in quantum gravity research.[52]

[45] Blau (2022), section 37.5, remark 1 below equation (37.62)

[46] Blau (2022) section 39.2.1

[47] Moschella (2005)

[48] Reviews of the geometry of de Sitter spacetime include Hartman (2017), Kim *et al* (2002), and Spradlin *et al* (2001).

[49] Spradlin *et al* (2001), equation (14)

[50] Kim *et al* (2002), section A.3

[51] Section 18

[52] Klemm and Vanzo (2004)

18 A cosmological event horizon

Consider the spatially flat case ($\epsilon = 0$), so the metric is

$$d\tau^2 = dt^2 - a^2(t)\,d\mathbf{x}^2.$$

By spherical symmetry, an object (or light pulse) whose initial velocity is in the x-direction will move in the x-direction forever, so we only need to consider the $1+1$-dimensional version

$$d\tau^2 = dt^2 - a^2(t)\,dx^2.$$

A null worldline satisfies $d\tau = 0$. In $1+1$ dimensions, a null worldline is automatically a geodesic, so the equation for a null geodesic is

$$\frac{dx}{dt} = \pm\frac{1}{a(t)}. \tag{44}$$

Now consider the more specific case $\epsilon = \rho = p = 0$ with $\Lambda > 0$. In this case, the Friedmann equations (33) and (34) have a solution of the form

$$a(t) = Ce^{\lambda t} \tag{45}$$

with λ defined by (43). This universe expands forever. Use (45) in equation (44) to deduce that the null geodesics are

$$x(t) = x(t_0) \pm \frac{e^{-\lambda t_0} - e^{-\lambda t}}{\lambda C},$$

which approaches the finite value $x(t_0) \pm e^{-\lambda t_0}/(\lambda C)$ as $t \to \infty$. This universe has a **cosmological event horizon**: a pulse of light from $x(t_0)$ will never reach points x with $|x - x(t_0)| > e^{-\lambda t_0}/(\lambda C)$.

In contrast, the universe described by (39) does not have an event horizon when $D \geq 2$. It expands forever, but it does not expand quickly enough to prevent a pulse of light that starts at $x(0)$ from reaching arbitrarily large values of $|x(t)|$.

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