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Covariant Derivatives and Curvature

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Abstract Partial derivatives of tensor fields are generally not tensor fields. The concept of a covariant derivative is a modification of the concept of a partial derivative, defined so that covariant derivatives of tensor fields are still tensor fields. Curvature tensors are defined in terms of covariant derivatives. General relativity is formulated with the help of a special covariant derivative that is metric-compatible and torsion-free.

Contents

1 Introduction

Article 33547 previewed the concept of a **geodesic**, which generalizes the idea of a "straight" path to curved spaces (or curved spacetimes). This article introduces the concept of a **covariant derivative**, which is the foundation for the concept of a geodesic. The concept of a covariant derivative is a modification of the concept of a partial derivative, and it allows writing the equations of motion in classical field theories (like general relativity) exclusively in terms of tensor fields, which avoids showing favoritism for any specific coordinate system (article 09894).

The first several sections introduce the concepts using a coordinate-free approach, and the remaining sections show how to express the same things in a coordinate representation.

2 Notation and prerequisites

This article assumes familiarity with the coordinate-free definitions and coordinate representations of tensor fields that were reviewed in article 09894. As in that article, a tensor field is said to be of type $\binom{k}{m}$ if its inputs are k one-form fields and m vector fields and its output is a scalar field.[1]

For clarity, this article uses different brackets (round and square) for different purposes. Square brackets are used to indicate grouping. Example: $x[y+z]$ means "first add y and z and then multiply the result by x." Round parentheses enclose the input to a function or map. Example: $\omega(V+W)$ is the result of using $V+W$ as the input to the map ω. Juxtaposition, writing symbols next to each other with no brackets or parentheses, denotes ordinary multiplication. One exception will be made: the result of applying a covariant derivative ∇_V to an object X will often be written $\nabla_V X$ instead of $\nabla_V(X)$.

[1] Lee (1997) uses the opposite notation, with meanings of the upper and lower integers exchanged.

3 The commutator of two vector fields

Recall that a **vector field** V is a linear **derivation** from the set of scalar fields to itself, so it satisfies the same product rule that derivatives satisfy:

$$V(fg) = V(f)g + fV(g) \tag{1}$$

for all scalar fields f, g. Consider two vector fields, V and W, applied in succession to a scalar field f:

$$V\big(W(f)\big).$$

The composition $V\big(W(\cdots)\big)$ is a map from scalar fields to scalar fields, but it doesn't qualify as a vector field because it's not a derivation (it doesn't satisfy the product rule):

$$V\big(W(fg)\big) \neq V\big(W(f)\big)g + f\,V\big(W(g)\big).$$

On the other hand, the **commutator** $[V, W]$ defined by

$$\boxed{[V,W](f) \equiv V\big(W(f)\big) - W\big(V(f)\big)}$$

does qualify as a vector field, because it is a derivation:

$$[V,W](fg) = [V,W](f)g + f[V,W](g)$$

for all scalar fields f, g. The commutator will become useful in section 13.

4 Covariant derivatives

A vector field acts like a derivative on scalar fields (equation (1)), but the action of a vector field on another vector field is undefined.[2] This section introduces the concept of a **covariant derivative** ∇_V, through which a vector field V can act on other vector fields (and, more generally, on other tensor fields) in a derivative-like way.[3]

For any given vector field V, the effect of ∇_V on another tensor field is required to satisfy the following conditions. When applied to a scalar field f, ∇_V is required to act just like the vector field V itself:

$$\nabla_V f \equiv V(f). \tag{2}$$

The effect of ∇_V on another vector field is required to have this derivative-like property when applied to a product fW, where f is a scalar field and W is a vector field:

$$\nabla_V(fW) = [\nabla_V f]W + f\nabla_V W. \tag{3}$$

The action of a covariant derivative on a 1-form field ω is determined by its action on scalar fields and vector fields through this condition:[4]

$$[\nabla_V\omega](W) \equiv \nabla_V\big(\omega(W)\big) - \omega\big(\nabla_V W\big). \tag{4}$$

The action of the covariant derivative on an arbitrary tensor field K is then given

[2] The naïve definition $V(W) = V^m(\partial_m W^k)\partial_k$ is not what we want, because it doesn't behave like the coordinate representation of a tensor field.

[3] The symbol ∇ here should not be confused with the ordinary gradient with respect to the "space" coordinates, for which some other articles in this series use the same symbol ∇.

[4] The notation $[\nabla_V\omega](W)$ means "apply ∇_V to ω to get the new one-form field $\nabla_V\omega$ and then use W as the input to this new one-form field."

by the similar condition

$$\boxed{\begin{aligned}[\nabla_V K](X_1,\, X_2,\, ...) \equiv \nabla_V\big(K(X_1,\, X_2,\, ...)\big) \\ - K\big(\nabla_V X_1,\, X_2,\, ...\big) \\ - K\big(X_1,\, \nabla_V X_2,\, ...\big) \\ - \cdots\end{aligned}} \tag{5}$$

where each argument X_n is either a vector field or a 1-form field, whichever is implied by the tensor field K. In words: the effect of ∇ on a tensor field K is given by its effect on the scalar field $K(X_1, X_2, ...)$ with its effects on the arguments X_n subtracted. Finally, when acting on any tensor field X, the covariant derivative is required to satisfy

$$\nabla_{fV+gW} X = f\nabla_V X + g\nabla_W X \tag{6}$$

for all scalar fields f, g and all vector fields V, W.

These conditions don't completely determine ∇, but after we specify its effect on vector fields, its effect on all other tensor fields is uniquely determined by these conditions.

5 Torsion and curvature

If the quantity

$$\nabla_V W - \nabla_W V - [V, W] \tag{7}$$

is not zero for some pair of vector fields V, W, then the covariant derivative is said to have non-zero **torsion**. Beware that

$$[\nabla_V W](f) \neq \nabla_V\big(W(f)\big).$$

In particular, the quantity

$$\nabla_V\big(W(f)\big) - \nabla_W\big(V(f)\big) - [V, W](f)$$

is always identically zero even though the quantity

$$\big[\nabla_V W - \nabla_W V - [V, W]\big](f)$$

is not necessarily zero (the torsion may be non-zero).

If the quantity

$$\big[\nabla_V \nabla_W - \nabla_W \nabla_V - \nabla_{[V,W]}\big]\,(U) \tag{8}$$

is non-zero for some triple of vector fields V, W, U, then the covariant derivative is said to have non-zero **curvature**.[5] This quantity may be non-zero even though the quantity

$$\big[\nabla_V \nabla_W - \nabla_W \nabla_V - \nabla_{[V,W]}\big]\,(f)$$

is identically zero for every scalar field f.

General relativity uses a covariant derivative whose torsion is zero, so the combination (7) is zero in general relativity. The curvature may still be non-zero.

[5]According to the proof of theorem 7.3 in Lee (1997), this is consistent with the way the word *curved* is used in articles 21808 and 48968.

6 Metric-compatible covariant derivatives

Given a metric field g, a covariant derivative is called **metric-compatible** if

$$\nabla_V g = 0$$

for all vector fields V. According to the definition (5), the preceding condition may also be written

$$\nabla_V\big(g(U,W)\big) = g\big(\nabla_V U, W\big) + g\big(U, \nabla_V W\big). \tag{9}$$

A metric-compatible derivative that also has zero torsion is called a **Levi-Civita connection**.[6] This is the covariant derivative used in general relativity. The next section shows that, given a metric field, the corresponding Levi-Civita connection is unique.

[6]The concept of a covariant derivative specializes the more general concept of a **connection**, which is a (path-dependent) way of comparing objects at different points of a manifold. The Levi-Civita connection is an example of a covariant derivative, so it is also an example of a connection (hence the name).

7 Uniqueness of the Levi-Civita connection

Given a metric field, the corresponding Levi-Civita connection satisfies this identity:

$$\begin{aligned} 2g\big(\nabla_V U, W\big) &= V\big(g(U,W)\big) + U\big(g(V,W)\big) - W\big(g(V,U)\big) \\ &\quad + g\big([V,U],W\big) - g\big([V,W],U\big) - g\big([U,W],V\big). \end{aligned} \tag{10}$$

To derive this, start by re-writing the first term on the right-hand side like this:

$$V\big(g(U,W)\big) = \nabla_V\big(g(U,W)\big) = g\big(\nabla_V U,\, W\big)\big) + g\big(U,\, \nabla_V W\big).$$

The first step holds because $g(U,W)$ is a scalar field (equation 2), and the second step holds because the covariant derivative is assumed to be metric-compatible (equation (9)). The first three terms on the RHS[7] of (10) may all be re-written this way, and the last three terms may be re-written using the zero-torsion condition like this (using one term as an example):

$$g\big([V,U],W\big) = g\big(\nabla_V U - \nabla_U V,\, W\big).$$

After re-writing things in this way, most terms on the RHS of (10) cancel, leaving only the remainder indicated on the LHS. This completes the derivation.

The right-hand side of (10) does not involve any covariant derivative, so it can be used to prove that the Levi-Civita connection is unique: use the ∇-independence of the right-hand side of (10) to see that (10) implies

$$g\big(\nabla_V U - \nabla'_V U,\, W\big) = 0$$

for any two Levi-Civita connections ∇ and ∇', and all vector fields W. Since g is a metric field, this implies $\nabla_V U - \nabla'_V U = 0$, so $\nabla = \nabla'$. This uniqueness result is called the Fundamental Theorem of Riemannian geometry.[8]

[7] RHS = right-hand side, LHS = left-hand side.

[8] Lee (1997), theorem 5.4

8 Coordinate representations

Recall (article 09894) that the coordinate representation of a vector field V is a combination of partial derivatives, $V = V^a\partial_a$, with smooth functions V^a as coefficients. According to the condition (2), the coordinate representation of a covariant derivative acting on a scalar field is

$$\nabla_V f = V^a\partial_a f. \tag{11}$$

In words: at any given point, the covariant derivative of f along V is just the partial derivative of f in the direction of the vector field V at that point.

The effect of covariant derivative on other tensor fields is uniquely determined after we specify its effect on vector fields. One way to do this is to choose a particular coordinate system and a three-index collection of functions Γ^b_{ac}, and then impose

$$\nabla_V W \equiv V^a\big[\partial_a W^b + W^c\Gamma^b_{ac}\big]\partial_b. \tag{12}$$

This clearly satisfies the constraints (3) and (6), so it qualifies as a covariant derivative. The **connection coefficients** Γ^b_{ac} are also called **Christoffel symbols**.[9] Every covariant derivative on a vector field can be written this way.[10]

The connection coefficients are different in different coordinate systems. The effect of a coordinate transformation on these coefficients may be deduced from the preceding equations. The result (not shown here) demonstrates that the connection coefficients Γ^b_{ac} are *not* the components of a tensor field.

Equation (12) says that the components of $\nabla_V W$ are $V^a\big[\partial_a W^b + W^c\Gamma^b_{ac}\big]$. We can also write this relationship as

$$\nabla_a W^b = \partial_a W^b + W^c\Gamma^b_{ac}. \tag{13}$$

This abbreviation can be misleading, because the left-hand side suggests that ∇_a is something that acts on individual compenents W^b, but the right-hand side demonstrates that it really involves *all* of W's components (a sum over the index c is

[9] Lee (1997), chapter 4, page 51, and Wald (1984), section 3.1, page 34. The name *Christoffel symbols* is often reserved for a special case, namely for the connection coefficients of the Levi-Civita connection (section 11).

[10] Lemma 4.4 in Lee (1997), or equations (3.1.7)-(3.1.15) in Wald (1984)

implied). The abbreviation is very convenient, though: it is standard in texts about general relativity, and I'll use it, too. Just remember that ∇ acts on the whole vector field, not on individual components.[11]

The coordinate representation of a 1-form is

$$\omega = \omega_a \, dx^a.$$

The effect of ∇_V on a 1-form can be derived using equations (2), (4), and (12). According to (2), the first term on the right-hand side of (4) is

$$\nabla_V\big(\omega(W)\big) = V\big(\omega(W)\big) = V^b \partial_b\big[\omega_a W^a\big].$$

According to (12), the second term on the right-hand side of (4) is

$$\omega\big(\nabla_V W\big) = \omega\Big(V^b\big[\partial_b W^a + \Gamma^a_{bc} W^c\big]\partial_a\Big) = V^b\big[\partial_b W^a + \Gamma^a_{bc} W^c\big]\omega_a.$$

When these are combined in (4), the terms involving $\partial_b W^a$ cancel each other, leaving

$$\nabla_V\big(\omega(W)\big) = V^b W^c\big[\partial_b \omega_c - \Gamma^a_{bc}\omega_a\big].$$

Therefore,

$$\nabla_V \omega = V^b\big[\partial_b \omega_c - \Gamma^a_{bc}\omega_a\big]\, dx^c. \tag{14}$$

This is the coordinate representation of the effect of ∇_V on a 1-form field. Equation (14) says that the components of $\nabla_V \omega$ are $V^b\big[\partial_b \omega_c - \Gamma^a_{bc}\omega_a\big]$. We can also write this relationship as

$$\nabla_a \omega_b = \partial_a \omega_b - \Gamma^c_{ab}\omega_c. \tag{15}$$

Notice that the Γ term has the opposite sign compared to equation (13).

The coordinate representation of the effect of ∇_V on other tensor fields may now be deduced from the condition (5). The result, expressed in words, is that the

[11] In the **abstract index notation**, the notation W^a represents the whole vector field, using the superscript as a "connector" to indicate how the vector field is used to construct other tensor fields. I'm not using that notation here.

effect of ∇_V on any tensor field consists of one partial-derivative term, plus one Γ-term for each of the tensor's superscripts, and minus one Γ-term for each of the tensor's subscripts.[12] The next section shows an example.

For a given vector field V, the covariant derivative ∇_V is a map from the set of tensor fields of type $\binom{k}{m}$ to itself. If the vector field V is left unspecified, as in equations (13) and (15), then ∇ converts a tensor field of type $\binom{k}{m}$ to a tensor field of type $\binom{k}{m+1}$, with an extra slot to accept the vector field V as an input specifying the direction of the derivative.

[12] Lee (1997), lemma 4.8

9 The product rule

Let $A^{\cdots}_{\cdots}$ be the components of a tensor field A, writing "$\cdots$" in place of a list of indices. Condition (5) implies the **product rule**[13]

$$\nabla_a[A^{\cdots}_{\cdots}B^{\cdots}_{\cdots}] = [\nabla_a A^{\cdots}_{\cdots}]B^{\cdots}_{\cdots} + A^{\cdots}_{\cdots}\nabla_a B^{\cdots}_{\cdots}. \tag{16}$$

This is an easy consequence of the result that was described in words at the end of section 8, which says that the left- and right-hand sides of (16) both have the same Γ-terms. After canceling the Γ-terms, the remainder is the familiar product rule for the partial derivatives ∂_a. The next section shows an example.

[13] Wald (1984) uses this as an axiom. This article didn't do it that way because I didn't want the definition to refer to components and I didn't want to use the abstract index notation.

10 Example: covariant derivative of a metric field

To illustrate the result that was described in words at the end of section 8, consider a metric field g. For any three vector fields V, U, W, the definition (5) gives

$$[\nabla_V g](U,W) = \nabla_V\big(g(U,W)\big) - g\big(\nabla_V U,\, W\big) - g\big(U,\, \nabla_V W\big).$$

Use equations (11), (13), and (15) to get the coordinate representation

$$(\nabla_a g_{bc})U^b W^c = \partial_a(g_{bc}U^b W^c) - g_{bc}(\partial_a U^b + \Gamma^b_{a\bullet}U^\bullet)W^c - g_{bc}U^b(\partial_a W^c + \Gamma^c_{a\bullet}W^\bullet).$$

(To make the pattern more evident, I'm using the symbol $\bullet$ for one of the indices.) This holds for all U, W, and ∂_a satisfies the familiar product rule, so it implies

$$\nabla_a g_{bc} = \partial_a g_{bc} - \Gamma^\bullet_{ab} g_{\bullet c} - \Gamma^\bullet_{ac} g_{b\bullet}. \tag{17}$$

In words: the effect of ∇_V on g_{ab} consists of one partial-derivative term minus one Γ-term for each of g_{ab}'s subscripts. This illustrates the general pattern described at the end of the previous section.

This result, together with equations (11), (13), and (15), immediately shows that ∇ satisfies the product rule when acting on $g_{bc}U^b W^c$.

11 The Levi-Civita connection coefficients

Equation (12) implies

$$\nabla_V W - \nabla_W V - [V,W] = \Big[\Gamma^b_{\ ac} - \Gamma^b_{\ ca}\Big] V^a W^c \partial_b.$$

This shows that a connection whose coefficients are symmetric in the two subscripts has zero torsion (and conversely). For the Levi-Civita connection, equation (17) implies

$$\partial_a g_{bc} = \Gamma^\bullet_{\ ab} g_{\bullet c} + \Gamma^\bullet_{\ ac} g_{b\bullet}. \tag{18}$$

By cyclically permuting the index-variables $a \to b \to c \to a$, we get these equivalent ways of writing (18):

$$\partial_b g_{ca} = \Gamma^\bullet_{\ bc} g_{\bullet a} + \Gamma^\bullet_{\ ba} g_{c\bullet}. \tag{19}$$

$$\partial_c g_{ab} = \Gamma^\bullet_{\ ca} g_{\bullet b} + \Gamma^\bullet_{\ cb} g_{a\bullet}. \tag{20}$$

Subtract equation (18) from the sum of equations (19) and (20) and use the symmetry of g and Γ in their subscripts to get

$$\partial_b g_{ca} + \partial_c g_{ba} - \partial_a g_{bc} = 2\Gamma^\bullet_{\ bc} g_{\bullet a},$$

and use the fact that the metric is invertible to get

$$\Gamma^a_{\ bc} = \frac{1}{2} g^{a\bullet}(\partial_b g_{c\bullet} + \partial_c g_{b\bullet} - \partial_\bullet g_{bc}). \tag{21}$$

These are the conection coefficients for the Levi-Civita connection.

12 Definition of the curvature tensor

Given a covariant derivative ∇, define the **curvature endomorphism** $R_{X,Y}$ by

$$R_{X,Y}Z \equiv \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \tag{22}$$

for all vector fields X, Y, Z. This satisfies[14]

$$R_{X,Y}(fZ) = f\,R_{X,Y}Z$$

for any scalar field f. Section 13 mentioned that the quantity (22) is related to the *curvature* of the covariant derivative ∇. More precisely, the **curvature tensor** is the tensor field R of type $\binom{1}{3}$ defined by the condition[15]

$$R(X,Y,Z,\omega) \equiv \omega(R_{X,Y}Z) \tag{23}$$

for all one-form fields ω.

[14] Lee (1997), exercise 7.1

[15] Proposition 7.1 in Lee (1997) shows that this does indeed give a tensor field as defined in article 09894.

13 Components of the curvature tensor

In a coordinate representation,

$$R(X, Y, Z, \omega) = R_{abc}{}^{d} X^a Y^b Z^c \omega_d.$$

The definition (22) clearly implies that the components $R_{abc}{}^{d}$ are antisymmetric in the first two subscripts: $R_{abc}{}^{d} = -R_{bac}{}^{d}$. To derive an explicit expression for the components, we can use a matrix notation in which Z^c are the components of a matrix $\mathbf{Z}$ of size $N \times 1$, and for each b, we can define a matrix $\mathbf{\Gamma}_b$ with components $[\mathbf{\Gamma}_b]_{ac} \equiv \Gamma^a_{bc}$ where the superscript a is the row index and the subscript c is the column index.[16] In particular, the components of the matrix product $\mathbf{\Gamma}_b\mathbf{Z}$ are $\Gamma^a_{bc}Z^c$. With this notation, the coordinate representation of equation (22) may be written

$$R_{X,Y} Z = \big[X^a(\partial_a + \mathbf{\Gamma}_a),\, Y^b(\partial_b + \mathbf{\Gamma}_b)\big]\mathbf{Z} = X^a Y^b \mathbf{R}_{ab}\mathbf{Z}$$

with[17]

$$\mathbf{R}_{ab} \equiv [\partial_a + \mathbf{\Gamma}_a,\, \partial_b + \mathbf{\Gamma}_b] \tag{24}$$

with $[A, B] \equiv AB - BA$, so the components of the curvature tensor are the components of the matrix $\mathbf{R}_{ab}$ for each a, b:[18]

$$R_{abc}{}^{d} \equiv [\mathbf{R}_{ab}]_{dc}. \tag{25}$$

More explicitly,

$$R_{abc}{}^{d} = \partial_a \Gamma^d_{bc} - \partial_b \Gamma^d_{ac} + \Gamma^d_{a\bullet}\Gamma^\bullet_{bc} - \Gamma^d_{b\bullet}\Gamma^\bullet_{ac} \tag{26}$$

with an implied sum over the index $\bullet$.

[16]The indices for the components of a *matrix* are all written as subscripts, using the standard convention in which the first index labels rows and the second index labels columns.

[17]Martin (1988), equation (6.14)

[18]Conventions for the overall sign and the order of the indices vary throughout the physics literature.

14 The Ricci tensor and curvature scalar

The **Ricci tensor** is[19]

$$R_{ab} \equiv R_{\bullet ab}{}^{\bullet} \tag{27}$$

with an implied sum over the index $\bullet$. Use equation (26) on the right-hand side of (27) to get[20]

$$R_{ab} = \partial_\bullet \Gamma^\bullet_{ab} - \partial_a \Gamma^\bullet_{\bullet b} + \Gamma^\bullet_{\bullet\times}\Gamma^\times_{ab} - \Gamma^\bullet_{a\times}\Gamma^\times_{\bullet b} \tag{28}$$

with implied sums over $\bullet$ and $\times$. The Ricci tensor is symmetric:

$$R_{ab} = R_{ba}.$$

The first, third, and fourth terms on the right-hand side of (28) are manifestly individually symmetric (given the symmetry $\Gamma^c_{ab} = \Gamma^c_{ba}$ that is evident in equation (21)), and the symmetry of the second term may be deduced from equation (21) with a little more effort.

The **curvature scalar** (or **scalar curvature** or **Ricci scalar**) is[21]

$$R \equiv g^{ab} R_{ab}. \tag{29}$$

If we take the covariant derivative ∇ to be the Levi-Civita connection, then the curvature scalar is a scalar field constructed entirely from the metric field. In general relativity, the equations governing the behavior the metric field come from the **Einstein-Hilbert action** $\int d^N x\ |\det g|^{1/2}\, R$, plus the action for the other fields. The curvature scalar R is the simplest nontrivial scalar field that can be constructed entirely from the metric field, so general relativity is the simplest generally covariant model[22] in which the behavior of the metric field is governed by a nontrivial condition.

[19] Don't confuse this with the matrix $\mathbf{R}_{ab}$ defined in equation (24).

[20] The overall sign of the relationship (28) appears to be standard the physics literature, despite variations in the conventions used for the curvature tensor (footnote 18).

[21] The overall sign of the relationship (29) appears to be standard the physics literature, despite variations in the overall sign used for the metric tensor itself. Section 15 will explain why this is useful.

[22] *General covariance* is defined in article 37501, which also mentions the reason for the factor $\sqrt{|\det g|}$.

15 Sign conventions

Conventions for curvature tensor vary,[23] both in the overall sign and the order of the indices, but the sign conventions used in equations (21), (28), and (29) appear to be standard in the literature about general relativity.[24] Those equations imply that the overall sign of the curvature scalar R depends on the sign convention used for the metric tensor g_{ab} (mostly-plus or mostly-minus), but the more important quantity in general relativity is the **Einstein tensor**

$$R_{ab} - \frac{1}{2} g_{ab} R,$$

and equations (28) and (29) ensure that the sign of this quantity doesn't depend the sign convention used for the metric tensor g_{ab}. Article 80838 explains why this is beneficial in the context of the gravitational field equation.

[23]The sign convention used in equations (24)-(25) avoids a minus sign in equation (23), and it agrees with the convention used in Lee (1997).

[24]Article 80838 quantifies this statement by citing several sources that all use equations (21), (28), and (29) even though they differ in their conventions for the curvature tensor.

16 Geodesics

Given a covariant derivative ∇, a vector field U is said to be **parallel transported** if

$$\nabla_U U \propto U. \tag{30}$$

Covariant derivatives are defined on vector (and tensor) *fields*, not on individual worldlines, but given a worldline $x^a(\lambda)$, we can think of its tangent vectors $\dot{x}^a \equiv dx^a/d\lambda$ as part of a spacetime-filling vector field U^a that is equal to $\dot{x}^a$ along the worldline.[25] In this case, the identity

$$U^a \partial_a U^b = \ddot{x}^a$$

holds everywhere along the worldline, where each overhead dot represents a derivative with respect to the parameter λ along the worldline. Using this in (30) gives

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c \propto \dot{x}^a. \tag{31}$$

Most worldlines don't satisfy this condition, because most worldlines are not parallel-transported along themselves with respect to the given covariant derivative ∇. A worldline that is parallel-transported along itself is called a **geodesic**. Given a geodesic, we can always parameterize it in such a way that the proportionality factor in equation (30) is zero. This is called an **affine parameterization**. When an affine parameterization is used, the geodesic equation (31) reduces to

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0. \tag{32}$$

Article 33547 highlights the geometric significance of this equation when the covariant derivative ∇ is taken to be the Levi-Civita connection corresponding to a given metric field.

[25] Wald (1984), page 34

17 A covariant version of integration-by-parts

In this section, ∇ denotes the Levi-Civita connection. Let $|\det g|$ denote the magnitude of the determinant of the matrix with components $g_{ab}(x)$, the components of the metric tensor. This section derives the identity[26]

$$\boxed{\partial_a\Big(\sqrt{|\det g|}\,V^a\Big) = \sqrt{|\det g|}\,\nabla_a V^a,} \tag{33}$$

which holds for the components V^a of any vector field V. Taken together with the product rule (equation (16)), this identity justifies an analog of integration-by-parts, but with covariant derivatives instead of ordinary derivatives. As an example, suppose $V^a = A^{ab}B_b$. Then equations (16) and (33) imply

$$\int d^Nx\,\partial_a\Big(\sqrt{|\det g|}\,V^a\Big) = \int d^Nx\,\sqrt{|\det g|}\,(\nabla_a A^{ab})B_b + \int d^Nx\,\sqrt{|\det g|}\,A^{ab}(\nabla_a B_b),$$

so if the left-hand side is zero, then the two terms on the right hand side are each other's negatives.

To derive the identity (33), start with this identity from article 18505:

$$\frac{\delta}{\delta M_{ab}}\det M = (\det M)(M^{-1})_{ab},$$

which implies

$$\partial_a \det M = (\det M)\sum_{b,c}(M^{-1})_{bc}(\partial_a M_{bc}).$$

This holds for any invertible matrix M whose components are functions of the coordinates. Specializing this to the matrix whose components are the components g_{ab} of the metric tensor gives

$$\partial_a|\det g| = |\det g|\,g^{bc}\partial_a g_{bc}, \tag{34}$$

[26] This is equation (3.4.10) in Wald (1984).

which implies

$$\partial_a\sqrt{|\det g|} = \frac{1}{2}\sqrt{|\det g|}\, g^{bc}\partial_a g_{bc}.$$

This implies

$$\partial_a\Big(\sqrt{|\det g|}\, V^a\Big) = \sqrt{|\det g|}\,\partial_a V^a + \frac{1}{2}\sqrt{|\det g|}\,(g^{bc}\partial_a g_{bc})V^a.$$

The explicit expression for Γ^c_{ab} in terms of the metric (section 21) implies

$$\Gamma^b_{ba} = \frac{1}{2}g^{bc}\partial_a g_{bc},$$

and using this in the preceding equation gives

$$\partial_a\Big(\sqrt{|\det g|}\, V^a\Big) = \sqrt{|\det g|}\,\big(\partial_a V^a + \Gamma^b_{ba}V^a\big).$$

On the right-hand side, the quantity in parentheses is the coordinate representation of $\nabla_a V^a$, so this completes the derivation of the identity (33).

18 References

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