Localized Operators as States on Boundaries

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Abstract In quantum theory, observables – things that are presumed to be measurable – are represented by linear operators on a Hilbert space, and states can be represented by elements of the Hilbert space. Quantum field theory (QFT) is a refinement of quantum theory in which each region of spacetime has an associated set of observables. The path integral formulation of QFT captures that association, and it also blurs the distinction between operators and states. An operator associated with a spacetime region R can be implemented by modifying the integrand of the path integral in a way that involves only the integration variables (field variables) in R. Evaluating the integrals over just those variables leaves a path integral with a state defined on the boundary of the now-excised region R. This leads to a way of thinking about QFT as a device that relates Hilbert spaces on different boundaries of spacetime to each other, sometimes called **functorial QFT**. This article introduces that way of thinking about QFT. The conventional concept of time evolution is a special case in which the path integral relates the Hilbert space associated with the initial time to the Hilbert space associated with the final time.

Contents

1 Introduction and example

In quantum theory, observables are represented by operators on a Hilbert space.[1] In quantum field theory (QFT), each region of spacetime has an associated set of observables.[2] This article reviews a refinement of that idea that occurs in the path integral formulation of QFT.

In the path integral formulation, operators are represented as modifications of the integrand of the path integral. The path integral involves many integration variables, each localized at a particular point in spacetime. An operator localized in a region R of spacetime may be represented as a modification of the integrand involving only the integration variables localized in R. When the integrand includes two or more such modifications corresponding to different operators localized in non-intersecting regions, the result of that composition depends on more than just the operators themselves. It might not even belong to the algebra generated by the two operators.

Here's an example that demonstrates that possibility.[3] Let C be a spacelike closed loop in d-dimensional spacetime, and let V be a $(d-1)$-dimensional ball that intersects C at a single point that is not on the boundary ∂V of V. Cut ∂V into two hemispheres H_1 and H_2. Let G be a compact Lie group and consider a model of a gauge field with gauged group G. Then we can choose a Wilson operator W localized on C and topological 't Hooft operators $T(H_1)$ and $T(H_2)$ localized on H_1 and H_2 with these properties:

$$T(H_1)T(H_2) = I \qquad\qquad T(H_1)\,W\,T(H_2) = zW \tag{1}$$

where I is the identity operator and z is a complex number that has magnitude 1 but is not equal to 1. These operators may all be represented as modifications of the integrand of the path integral. Let A be the combination of the modifications

[1] Article 03431

[2] Article 21916

[3] This example uses a one-form symmetry called **center symmetry**. Article 09181 introduces the general concepts. Article 82508 derives equations (1).

representing $T(H_1)$ and $T(H_2)$, and let B be the modification representing $W+W^2$. The first equation in (1) says that the modification A by itself represents the identity operator, but the second equation in (1) says that the operator represented by the composition of A and B is $T(H_1)(W+W^2)T(H_2)$, which is proportional to $W+zW^2$. This shows that the operator represented by the composition of A and B does not belong to the algebra generated by the operators they individually represent.

We can think of A as representing the identity operator I, but we can also think of A as representing an *endomorphism* of the operator algebra[4] – a map from the algebra of operators to itself instead of from the Hilbert space to itself. Even this doesn't fully capture what A is, though. To appreciate why, recall the time-slice principle:[5] the operator $W+W^2$ that we previously viewed as being localized on C may also be viewed as localized in a neighborhood of any Cauchy surface far in the future of C and H_1 and H_2. This implies that the same operator may be represented as a (very complicated)[6] modification B' of the integrand that only involves field variables in the far future. The operator represented by the composition of A and B' just $W+W^2$, which is different than the operator $zW+z^2W^2$ represented by the composition of A and B, even though the operators represented by B and B' by themselves are equal to each other (both equal to $W+W^2$).[7]

The message is that in the path integral formulation, A and B are not just operators (or just endomorphisms). They are something more. This article calls them *modifiers*. The name is not standard, but the concept underlies many recent developments in QFT. This article reviews the fact that a modifier can be fully represented by excising the region of spacetime in which the operator is localized and replacing it with a state on the boundary of that region, an idea that can be formalized without relying on path integrals.[8]

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[4] Benedetti *et al* (2025), section 3.2

[5] Article 21916

[6] *Topological operators* are special because they have simple descriptions in many different regions (article 82508).

[7] The *localized morphism* defined in Haag (1996), text after equation (IV.2.7) (also called a *localized endomorphism* in Halvorson and Müger (2006), definition 8.1) doesn't affect any operators localized in the causal complement of a given region, but it's still just an ordinary endomorphism: its effect on a given operator depends only on that operator, not on any additional information.

[8] Section 16

2 Outline

- Sections 4-5 review the structure of the path integral.
- Sections 6-7 introduce the idea of states on arbitrary boundaries.
- Sections 8-9 review the localized operator concept without using the path integral formulation.
- Sections 10-11 introduce the idea that a modification of the integrand of the path integral is more than just an operator.
- Sections 12-14 review how such a modification may be represented as a state on the boundary of the region in which the operator is regarded as being localized.
- Sections 15-16 mention formalizations of that idea that don't rely on the path integral formulation.

3 Notation and conventions

In this article, **operator** means a linear operator on the Hilbert space.[9] Many sources (including other articles in this series) also use the word *operator* for what this article calls a *modifier*, which is more than just a linear operator on the Hilbert space.[10] Notation:

- d is the number of dimensions of spacetime.
- X is a lower-dimensional submanifold of spacetime.
- M, R, S are d-dimensional regions in spacetime.
- ∂M is the boundary of M.
- If $X \subset M$, then $M \setminus X$ is what remains after removing X from M.
- ϕ is a generic set of field variables in the context of a path integral.
- $[\phi]_M$ is the set of field variables localized in M.
- Ψ is a function of some of the field variables ϕ used to represent a state in the path integral formulation.
- $\text{Action}[\phi]$ is the action functional.
- i is a square root of -1.
- $\mathcal{A}(R)$ is the algebra of operators localized in R.
- $\mathcal{M}(R)$ is the set of *modifiers* localized in R (section 10).
- $\tau(A, B)$ is the composition of modifiers A and B (section 11), also called the time-ordered product in situations where that name makes sense. AB is the algebraic product (an operator) of A and B when they are regarded as mere operators.

[9] Article 74088

[10] Section 2

4 Time evolution and path integrals

In QFT, a **path integral** is an integral over **field variables**, each of which is associated with a specific element (like a point or a link) of discrete spacetime.[11] The path integral version describing the evolution of a state from time t to time t' has the form[12]

$$\Psi'[\phi]_{t'} \propto \int_{<t'} [d\phi]\; e^{i\,\text{Action}[\phi]}\Psi[\phi]_t \tag{2}$$

where

- Ψ and Ψ' are the initial and final states,
- $[\phi]_t$ denotes the set of field variables associated with time t,
- the **action**, denoted $\text{Action}[\phi]$, is a function of all the field variables from times t to t' inclusive,
- the integral is over of the field variables associated with times in the range $\geq t$ and $< t'$.

The details of the action depend on the model. In this article, the important property of the action is that it is a sum of terms that each depend only on field variables associated with a small neighborhood of a point in spacetime.[13]

[11] Article 46333 describes a relatively general way to discretize spacetime. This article uses some of the language (like *link*) introduced there.

[12] Other articles describe this in more detail: article 63548 for a model whose only field is a scalar field, article 89053 for a model whose only field is a gauge field.

[13] The path integral is often defined with the help of **Wick rotation** (footnote 12). The ideas in this article don't directly depend on understanding Wick rotation.

5 States as functions of field variables

Each state in equation (2) is a function of the field variables associated with a single time. Elements of the Hilbert space are complex-valued functions $\Psi[\phi]$ of the field variables at a single (unspecified) time,[14] with the inner product between two states Ψ_1 and Ψ_2 defined by

$$\int [d\phi]\ \Psi_1^*[\phi]\Psi_2[\phi].$$

Linear operators on the Hilbert space are typically represented as differential operators acting on the functions $\Psi[\phi]$. Such representations may involve derivatives with respect to the field variables and multiplication by functions of the field variables.[15] The inner product is used to determine the corresponding representation of the operator's adjoint.

The rest of this article does not use the inner product.[16]

[14]This assumes that the spacetime manifold is homeomorphic to $\mathbb{R} \times M_s$ and that the "time" manifold $\mathbb{R}$ and "space" manifold M_s are separately discretized so that the definition of the Hilbert space doesn't need to specify the time with which the two states are both associated. In principle, though, we could define a different Hilbert space for each time. The idea that will be introduced in section 6 exploits this.

[15]Article 52890 illustrates this for a scalar field.

[16]Witten (2025) shares insights about how Wick rotation interacts with the definition of the inner product.

6 States on arbitrary boundaries

Let R be a d-dimensional proper submanifold of d-dimensional spacetime,[17] not necessarily connected. Let S be another d-dimensional proper submanifold, also not necessarily connected, and suppose the intersection $R \cap S$ is empty. Figure 1 illustrates the setup.

Let M be what remains of spacetime after deleting the interiors of R and S (retaining their boundaries ∂R and ∂S), and consider the path integral

$$\Psi'[\phi]_{\partial R} \propto \int_{M\setminus\partial R} [d\phi]\; e^{i\,\text{Action}[\phi]_M}\,\Psi[\phi]_{\partial S} \tag{3}$$

where

- $[\phi]_X$ denotes the set of field variables associated with a submanifold X,
- Ψ and Ψ' are complex-valued functions of the indicated field variables,
- $\text{Action}[\phi]_M$ is the part of the action that remains after discarding terms involving field variables outside M,
- the integral is over of the field variables associated with $M \setminus \partial R$ (the field variables in M but not in ∂R).

To make this precise, we would need to specify exactly how ∂R, ∂S, $[\phi]_X$, and $\text{Action}[\phi]_M$ are defined, given that spacetime is discretized.[18] Section 7 will partially address this.

Equation (2) is a special case of (3) in which S is the region before (and including) the initial time t and R is the region after (and including) the final time t', but equation (3) is much more general. In this article, R will always be the future of a given Cauchy hypersurface, so $\Psi'[\phi]_{\partial R}$ will always be the final state in the usual sense, but S may have a more elaborate topology.

[17] Article 09181 defines **proper submanifold**. Roughly: a proper submanifold includes its own boundary.

[18] Example: if the action has a term $(\phi_1-\phi_2)^2 = \phi_1^2+\phi_2^2-2\phi_1\phi_2$, then does "discarding terms involving ϕ_2" mean that we should discard $(\phi_1-\phi_2)^2$, or should we retain ϕ_1^2?

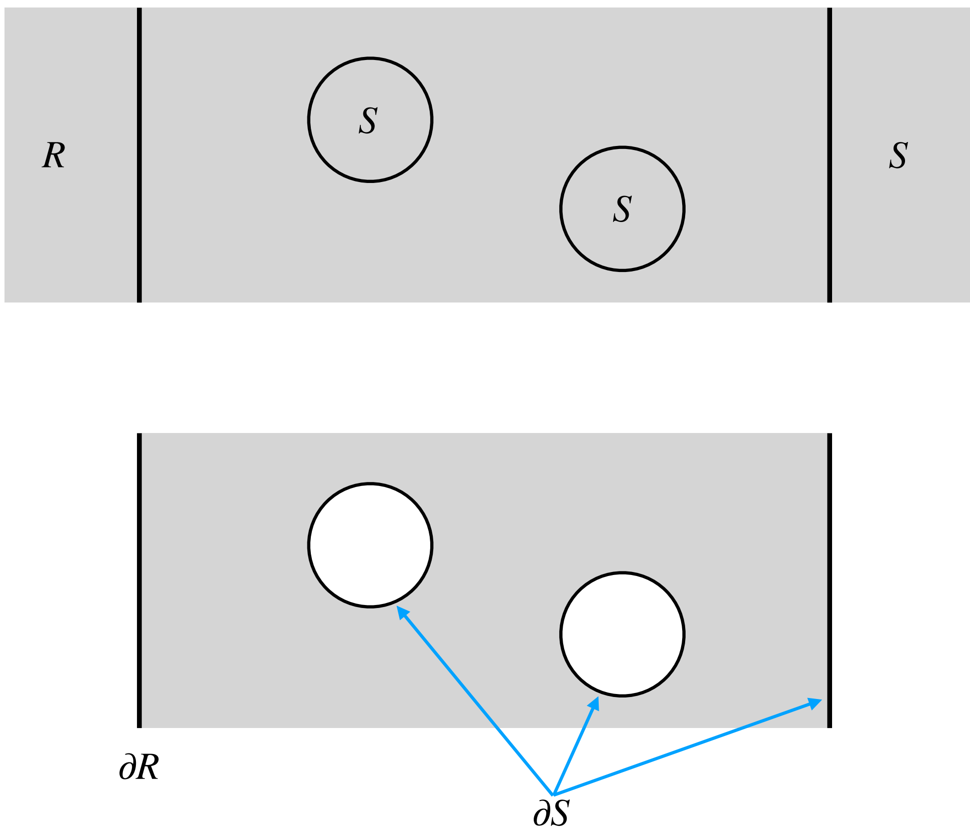

Figure 1 – In these pictures, spacetime is two-dimensional. The top picture shows an example of spacetime regions R and S that can be used in equation (3). R is the region to the left of the left vertical line. S has three parts: one to the right of the right line, and the interiors of the two circles. The bottom picture shows what remains after excising R and S. The integration variables in equation (3) live in the shaded region in the bottom picture and on the boundary ∂S of S. The resulting state is a function of the variables that live on ∂R.

7 Gluing path integrals together

Let M be a d-dimensional spacetime manifold whose boundary B may have one or more connected components. Choose a $(d-1)$-dimensional submanifold C that cuts M into two parts, M_+ and M_-, both defined to include C, and suppose C does not intersect B. Let $B_\pm$ be the parts of B that belong to $M_\pm$ (figure 2). Consider the two path integrals

$$\Psi_0[\phi]_C \propto \int_{M_-\backslash C}[d\phi]\; e^{i\,\text{Action}[\phi]_{M_-}}\Psi_-[\phi]_{B_-} \tag{4}$$

$$\Psi_+[\phi]_{B_+} \propto \int_{M_+\backslash B_+}[d\phi]\; e^{i\,\text{Action}[\phi]_{M_+}}\Psi_0[\phi]_C. \tag{5}$$

In words:

- The first one ingests a specified state on B_- and produces a state on C.
- The second one ingests a specified state on C and produces a state on B_+.

Using the state produced by the first path integral as the specified state in the second path integral gives the combined path integral

$$\Psi_+[\phi]_{B_+} \propto \int_{M\backslash B_+}[d\phi]\; e^{i\left(\text{Action}[\phi]_{M_+}+\text{Action}[\phi]_{M_-}\right)}\Psi_-[\phi]_{B_-}, \tag{6}$$

which ingests a specified state on B_- and produces a state on B_+. For (6) to be consistent with (3), the right side must be equal to

$$\int_{M\backslash B_+}[d\phi]\; e^{i\,\text{Action}[\phi]_M}\Psi_-[\phi]_{B_-},$$

which implies

$$\text{Action}[\phi]_{M_+} + \text{Action}[\phi]_{M_-} = \text{Action}[\phi]_M.$$

This is a condition on the definition of $\text{Action}[\phi]_X$ that section 6 loosely described as "discarding terms involving field variables outside X."

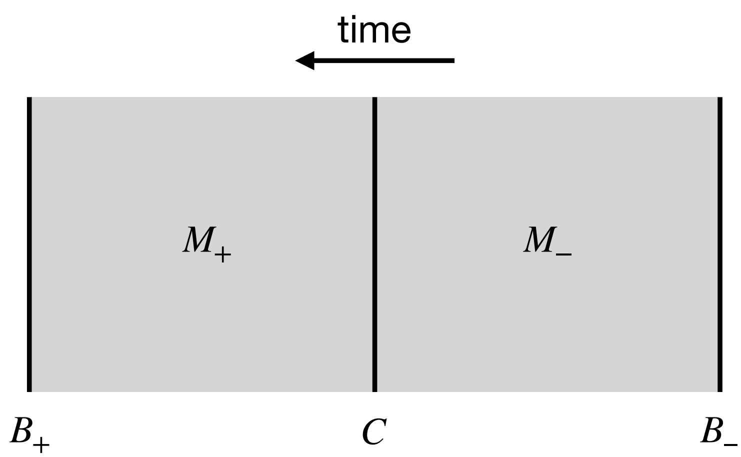

Figure 2 – This picture shows an example of a guling arrangement described in section 7. Time runs from right to left. Starting with the path integral from B_- to B_+, cutting the path integral along C gives two path integrals, one from B_- to C and one from C to B_+.

8 Localized operators

This section reviews the concept of associating observables with regions of spacetime in QFT. Section 10 will review how the path integral automatically accounts for this association.

In quantum theory, observables are represented by operators on a Hilbert space.[19] Quantum field theory is a refinement of quantum theory in which each d-dimensional submanifold of d-dimensional spacetime has an associated set of observables.[20,21,22] In this article, $\mathcal{A}(R)$ denotes the algebra generated by operators that represent observables associated with a spacetime region R. This is called a **local algebra**. Most of the operators in $\mathcal{A}(R)$ don't represent observables, but every operator in $\mathcal{A}(R)$ will be called **localized in** R.[23]

The same operator may be localized in multiple regions of spacetime: two local algebras $\mathcal{A}(R_1)$ and $\mathcal{A}(R_2)$ may share some operators with each other even if R_1 and R_2 don't intersect each other. The identity operator is a trivial example: it belongs to the algebra $\mathcal{A}(R)$ for every R. Another supply of examples is given by *topological operators*.[24] A wealth of examples is supplied by the **time-slice principle**,[20] which says that if R_1 is a neighborhood of one Cauchy hypersurface[25] and R_2 is a neighborhood of another Cauchy hypersurface, then $\mathcal{A}(R_1) = \mathcal{A}(R_2)$. The important message is that the region in which an operator is localized is additional information that cannot be inferred from the operator itself.

[19] Article 03431

[20] Article 21916

[21] This way of expressing the definition uses the Heisenberg picture.

[22] One important example of a d-dimensional submanifold is a d-dimensional open ball, but the submanifold may also have a more interesting topology, like a neighborhood of a loop.

[23] The net of algebras $\mathcal{A}(R)$ is not necessarily additive. Article 21916 (version dated 2025-07-25 or later) explains what this means.

[24] Article 09181

[25] Section 3.1 in Witten (2019) defines **Cauchy (hyper)surface**.

9 Time-ordered products

Let R_A and R_B be non-intersecting neighborhoods of two non-intersecting Cauchy hypersurfaces, and let A and B be operators localized in R_A and R_B, respectively. In this simple situation, the **time-ordered product** is defined to be[26]

$$\tau(A,B) \equiv \begin{cases} AB & \text{if } R_A \text{ is in the future of } R_B, \\ \pm BA & \text{if } R_B \text{ is in the future of } R_A. \end{cases} \tag{7}$$

More generally, if A is localized in R_A and B is localized in R_B, equation (7) becomes

$$\tau(A,B) \equiv \begin{cases} AB & \text{if } R_A \text{ does not intersect the causal past of } R_B, \\ \pm BA & \text{if } R_B \text{ does not intersect the causal past of } R_A. \end{cases} \tag{8}$$

This assumes **microcausality**, which says $AB = \pm BA$ if neither R_A nor R_B intersects the other's causal past. If they both intersect each other's causal past, then $\tau(A,B)$ is undefined.

Unlike the ordinary algebraic product of two operators,[27] the time-ordered product relies on information about where in spacetime those operators are localized. This extra information is essential, because the region of spacetime at which an operator is localized cannot be inferred from the operator itself.[28] We can think about $\tau(A,B)$ as having only two inputs A, B if we think of A and B themselves as carrying that extra information. Then A and B are more than just linear operators on a Hilbert space. Sections 10-11 will review how the path integral formulation captures this extension of the concept of an "operator."

[26] The sign is negative if A and B both have odd fermion grade and is positive if either one has even fermion grade.

[27] In this article, an **operator** is a linear transformation of a Hilbert space (section 10). The ordinary algebraic product of two operators A and B is defined by applying both linear transformations in a specified order.

[28] Section 8

10 Operators as modifications of the integrand

In the path integral formulation, field variables are integration variables indexed by elements of discrete spacetime (when spacetime is discretized to define the path integral unambiguously). Given a subset X of spacetime, an operator localized on X is typically described by modifying how the integrand of the path integral depends on the field variables indexed by X.[29] Such modifications of the integrand of the path integral are often called **(operator) insertions**[30] or **defects**,[31] depending on the author and/or on the nature of the modification. To avoid any specific connotations that might be conveyed by the other names, this article will call it a **modifier**.[32] The set of modifiers localized in R will be denoted $\mathcal{M}(R)$.

A modifier is more than just an operator on the Hilbert space of initial states. In particular, the information that defines a modifier includes a region of spacetime in which it is localized. In the example in section 9, the time-ordered product of A and B would be undefined if A, B were viewed as nothing more than operators on the Hilbert space of initial states, but it is well-defined if they are viewed as modifiers. Section 2 showed that specifying an operator together with a localization region still isn't sufficient: a modifier is more than this.

A map is called *forgetful* if it simply discards some of the information in the definition of one thing to reduce it to the definition of a simpler thing. Forgetful maps occur throughout mathematics. The case of interest in this article is the forgetful map $\mathcal{M}(R) \to \mathcal{A}(R)$ that converts modifiers to mere operators. To keep the notation light, this article uses the same symbol for a modifier $A \in \mathcal{M}(R)$ and the operator $A \in \mathcal{A}(R)$ to which it reduces under the forgetful map.

[29] Article 63548 describes basic examples. The articles cited in article 22721 describe several more examples.

[30] Copetti *et al* (2025)

[31] Iqbal (2024)

[32] This name is not standard.

11 The composition of modifiers

Consider two modifiers $A \in \mathcal{M}(R_A)$ and $B \in \mathcal{M}(R_B)$. If R_A and R_B do not intersect each other, then applying both modifiers to the integrand of the path integral defines[33] another modifier $\tau(A, B) \in \mathcal{M}(R_A \cup R_B)$ that we may call the **composition of modifiers**. This generalizes in the obvious way to the composition of any number of modifiers $A, B, ...$, denoted $\tau(A, B, ...)$.

If the regions R_A and R_B do not both intersect each other's causal past, then the composition $\tau(A, B)$ reduces to the time-ordered product of A and B (section 9). In that case, the name *time-ordered product* is appropriate because $\tau(A, B)$ is equal to either AB or $\pm BA$, where juxtaposition denotes the ordinary algebraic product of A and B when they are regarded as mere operators.

If the regions R_A and R_B do both intersect each other's causal past, then the name *time-ordered product* seems less appropriate:[34] the "time order" is undefined, and the result $\tau(A, B)$ is not necessarily an algebraic product of A and B. It might not even belong to the algebra generated by $A \in \mathcal{A}(R_A)$ and $B \in \mathcal{A}(R_B)$.[35] This article uses the less presumptuous name *composition of modifiers* instead.[36]

[33]In a model with fermion fields, a sign ambiguity may exist because the field variables themselves may be anticommuting. This article ignores that complication.

[34]If $A = \sum_j A_j$ and $B = \sum_k B_k$ for operators A_j and B_k that are all localized in non-overlapping time intervals, then $\tau(A, B)$ can be defined as $\sum_{j,k} \tau(A_j, B_k)$. Even in that case, though, the composition $\tau(A, B)$ is using extra information about how A and B are constructed, not just the operators A and B and their localization regions.

[35]Section 2 described an example.

[36]This name is not standard.

12 Modifiers as states on new boundaries

Consider the path integral of a lattice QFT with only scalar fields. Let R be a region of spacetime with no sites on its boundary, as depicted in figure 3, and let $\phi(R)$ be the set of field variables associated with the sites in R. Let ∂R be the set sites that are just inside R and have immediate neighbors just outside R. Consider a modifier localized in $R \setminus \partial R$ that is represented by inserting a function of the field variables $\phi(R \setminus \partial R)$ into the integrand of the path integral. Doing the path integral over the field variables $\phi(R \setminus \partial R)$ eliminates those variables but leaves a new function $\Psi[\phi(\partial R)]$ in the integrand.[37] As far as modifiers localized elsewhere are concerned, this function $\Psi[\phi(\partial R)]$ is an equivalent representation of the original modifier.

Conversely, we can excise (delete) any region $R \setminus \partial R$ of spacetime and impose a state on the boundary ∂R of the excised region. This provides many new modifiers – and therefore operators – that we might not have considered otherwise. As an example, we could choose the function $\Psi[\phi(\partial R)]$ to be nonzero only for specific values of the field variables $\phi(\partial R)$, which is the same as imposing those values as boundary conditions on the fields in the path integral. This suggest a way to formalize the concept of a *modifier* in a way that doesn't rely on the path integral formulation. Section 16 will cite some references.

[37]In a lattice QFT with only gauge fields, the same idea works if we define $\Phi(R)$ to be the set of link variables that have at least one endpoint in R and define $\Phi(\partial R)$ to be the set of link variables that have *only* one endpoint in R. The resulting state is necessarily gauge invariant. To deduce this, consider only the part of the path integral (with the part of the action) that involves variables in $\Phi(R)$. The integrand (including the action) is a gauge-invariant function of all of the link variables, so integrating over any subset of the link variables must leave a gauge-invariant function of the remaining ones.

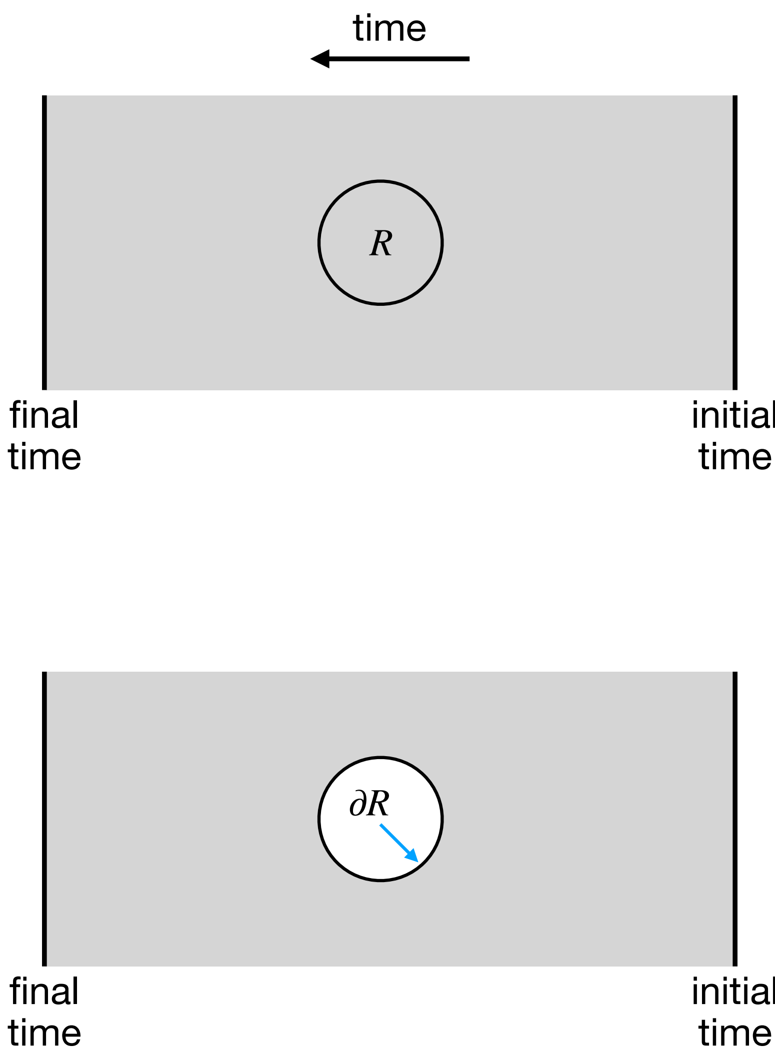

Figure 3 – These pictures illustrate the setup for describing a modifier localized in R as a state on the boundary ∂R of R. The interior of R is excised in the sense that the path integral no longer involves any field variables localized in R. The integrals over those field variables were already evaluated to produce the state on ∂R. Segal (2021) uses a similar picture to convey the same idea (timestamps 7:13 to 8:28) but with the goal of developing an axiom system that doesn't rely on path integrals.

13 The excised region as keep-out zone

When a modifier is constructed as described in section 12, a region of spacetime is effectively excised, preventing the construction of other modifiers whose localization regions intersect the excised region. Such a **keep-out zone** would typically exist anyway, because the composition is typically defined only for modifiers whose localization regions don't intersect each other. If they do intersect, then the prescription in section 11 to "apply both modifications to the integrand" might be undefined.

Many operators of interest in studies of QFT are nominally localized on lower-dimensional submanifolds of d-dimensional spacetime – on points, curves, surfaces, and so on. In the continuum limit, observables in d-dimensional spacetime strictly localized on lower-dimensional manifolds like points or curves typically don't exist.[38] That's not a problem for physics, because real experiments don't have unlimited resolution anyway. For many purposes, though, pretending that observables can be strictly localized on such lower-dimensional manifolds can simplify the equations and the intuition. We can get away with this as long as we are careful to avoid multiplying observables whose lower-dimensional localization regions intersect each other, so each operator's localization region is once again a keep-out zone for other operators.[39,40] If we're imposing that constraint anyway, then we don't lose anything by constructing the operators (or modifiers) as described in section 12: in the path integral, excise (delete) an arbitrarily small d-dimensional neighborhood of the given lower-dimensional manifold from d-dimensional spacetime and impose an appropriate state on the boundary of the excised region.

[38] Article 09181

[39] Example: we can treat the scalar field operator $\phi(x)$ as though it were strictly localized at the point x, as long as we never set $y = x$ in products like $\phi(x)\phi(y)$.

[40] The region over which one operator is smeared (article 10690) should not intersect the keep-out zones of other operators. This avoids the issue mentioned in article 10690 about interference between smearing and the composition of modifiers (called the *time-ordered product* in that article).

14 The importance of entanglement

Consider a Cauchy hypersurface X at a time intermediate between the initial and final times. Let R be a neighborhood of the intermediate Cauchy hypersurface, and suppose that R does not touch the initial or final times. The boundary ∂R has two components: one slightly before X, and one slightly after X. Denote these components by $(\partial R)_-$ and $(\partial R)_+$, respectively (figure 4).

Consider the path integral obtained from (2) by applying an modifier somewhere in R. The path integral covers the part of spacetime from the initial time to the final time. Now evaluate the integrals over all the field variables in R, which induces a state on ∂R.[41] The new path integral only covers the parts of spacetime from the initial time to $(\partial R)_-$ and from $(\partial R)_+$ to the final time. These two parts of spacetime are not connected to each other, but the state at the final time is still affected by the state at the initial time, just like it was before we evaluated the integrals in R. This works because the induced state on $\partial R = (\partial R)_- \cup (\partial R)_+$ does not factorize: it cannot be written as the product of a state on $(\partial R)_-$ and a state on $(\partial R)_+$.[42]

This non-factorization property could be described as **entanglement** between $(\partial R)_-$ and $(\partial R)_+$. That word is used more often when the two entangled parts of the system are separated from each other along a spacelike direction. In this example they are separated from each other along a timelike direction instead, but it's the same idea mathematically. In a relativistic model where Wick rotation can be used to remove the distinction between timelike and spacelike directions, this example can be turned into a proof that all states with finite (non-infinite) energy are entangled with respect to location in space.[43]

[41] Section 12

[42] For a specific example, think of the free scalar field ϕ described using the path integral formulation in article 63548. The action has terms of the form $(\phi(x)-\phi(y))^2$ where x and y are adjacent points in the lattice. Evaluating an integral of the form $\int d\phi(y)\ \exp\big(c(\phi(x)-\phi(y))^2+c(\phi(y)-\phi(z))^2\big)$ with $c\neq 0$ gives a function of $\phi(x)$ and $\phi(z)$ that cannot factorized into the product of a function of $\phi(x)$ and a function of $\phi(z)$.

[43] Article 00980 illustrates this for a free scalar quantum field.

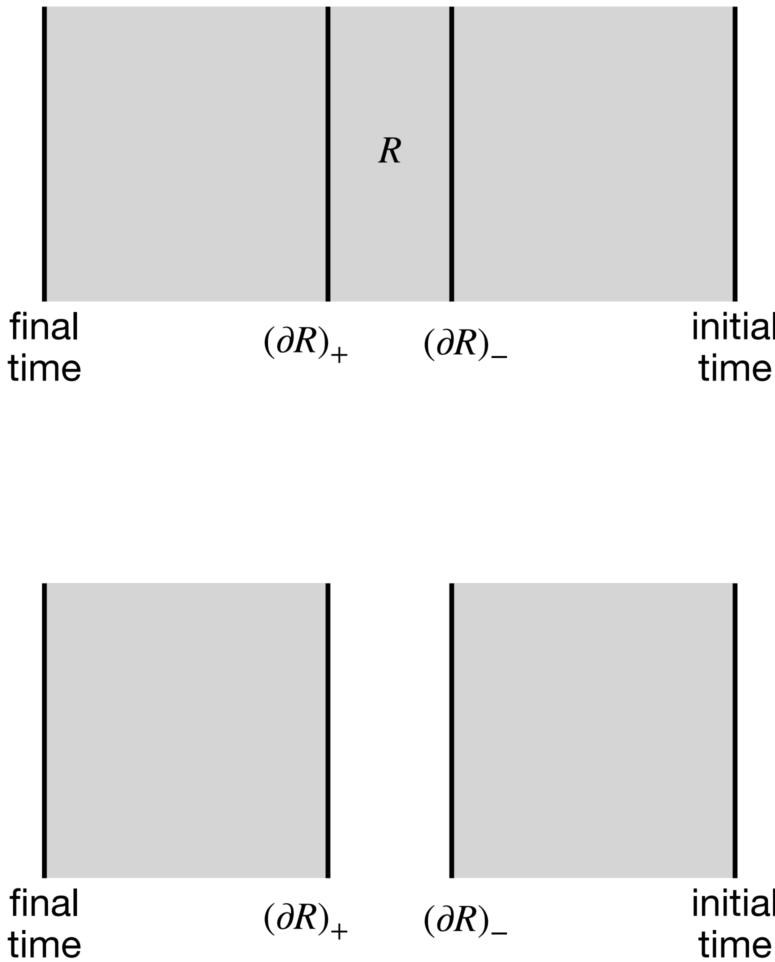

Figure 4 – These pictures illustrate the setup described in section 14. R is a neighborhood of a Cauchy hypersurface X. (X is not highlighted.) Its boundary has two components, $(\partial R)_+$ and $(\partial R)_-$, that lie on opposite sides of X. Integrating over the field variables in R leaves a state on $(\partial R)_+ \cup (\partial R)_-$. This state cannot be written as a product of states on $(\partial R)_+$ and $(\partial R)_-$, because if it could then the state at the final time would be independent of the state at the initial time, which would violate the principle that time evolution should be unitary.

15 The state-operator correspondence in CFT

Section 12 described a correspondence between operators localized in R and states on ∂R. Each operator localized in R is represented by such a state (by construction), and each state on ∂R may be interpreted as such an operator (by definition). The name *state-operator correspondence* is usually reserved for something more specific – to a special property of conformal field theory (CFT).[44] This section explains how that special property relates to the correspondence in section 12.

In smooth spacetime, relativistic QFT has the **Reeh-Schlieder property**.[45] To describe this property in the path integral formulation, let X_i be an initial Cauchy hypersurface, let X_f be a final Cauchy hypersurface, and let R be an arbitrarily small neighborhood of a point between X_i and X_f. Roughly, the Reeh-Schlieder property says that all the states in the Hilbert space associated with X_f may be obtained by choosing a single state on X_i and applying operators localized in R. The **state-operator correspondence** in CFT comes from using conformal symmetry to relate X_f to ∂R, so the two-way correspondence between operators in R and states on ∂R becomes a two-way correspondence between operators in R and states on X_f, like a two-way version of the Reeh-Schlieder property.[46]

The Reeh-Schlieder property and the (CFT) state-operator correspondence both rely on smooth spacetime. The correspondence described in section 12 does not: it works even if spacetime is discretized to make the path integral unambiguous.

[44] Most models don't have conformal symmetry, and models in discretized spacetime never do except in the low-resolution limit at a critical point (article 10142).

[45] Article 00980

[46] The fact that the state-operator correspondence in CFT is related to the Reeh-Schlieder property is mentioned in `https://ncatlab.org/nlab/show/Reeh-Schlieder+theorem`.

16 From path integrals to axioms

This article explained how the path integral formulation of QFT may be viewed as a device that relates states on one set of components of the boundary of a spacetime manifold to states on another component of the boundary. One idea for a axiomatic characterization of QFT tries to capture that view in abstract terms – without referring to the path integral construction – by describing QFT as a functor from a category of spacetimes-with-boundaries (bordisms) to a category of Hilbert spaces. Sometimes the idea is called **functorial QFT**.[47] In that approach, a localized observable can be described as explained in section 12 – by excising the region of spacetime where it's localized and imposing an appropriate state on the boundary of the excised region.[48] This is an abstract way of defining what section 10 called a *modifier*. Axiom systems of this kind is relatively well-developed for conformal QFT (CFT) and topological QFT (TQFT).[49] The idea is still relatively exploratory for quantum field theory in general,[50,51,52] but it is widely used in the study of anomalies.[53]

[47]Dedushenko (2022), section 2.3; Schreiber (2008)

[48]Section 2.2 in Tachikawa (2017) describes this for conformal QFT, using the state-operator correspondence. Schreiber (2008) uses heavy category-theoretic language to address it for general QFT. Section 3 in Kontsevich and Segal (2021) (also Segal (2021), timestamps 7:10 to 8:30) describes the idea for operators nominally localized at individual points.

[49]Tachikawa (2017), section 2.1; Schreiber (2008), section 1

[50]Dedushenko (2022), section 2.3

[51]An earlier rendition of the idea was given in Dijkgraaf and Witten (1990), section 6.1.

[52]Choosing such a system of axioms amounts to deciding precisely what *quantum field theory in general* should mean. Section 3 in Kontsevich and Segal (2021) says "The guiding principle of this approach is to preserve as much as possible of the path-integral intuition." An earlier account of this proposed definition of QFT is given in Segal (2011), timestamps 4:20 to 9:05.

[53]Monnier (2019), section 2; Freed (2014), section 2.1; Monnier (2015), section 2.1; Monnier (2014)

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