The Free Scalar Quantum Field: Vacuum State

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Abstract Article 52890 explained how to construct models of scalar quantum fields in a mathematically straightforward way by treating space as a lattice. The *free* scalar field, whose equation of motion is linear, is a special case in which the energy eigenstates can all be determined explicitly. This article uses the free scalar field to highlight some basic points about the **vacuum state**, the state with the lowest energy.

Contents

1 Introduction

Article 52890 showed how to construct a model of a single scalar quantum field $\phi(\mathbf{x}, t)$ whose time dependence is governed by the equation of motion[1]

$$\ddot{\phi}(\mathbf{x}, t) - \nabla^2\phi(\mathbf{x}, t) + V'\big(\phi(\mathbf{x}, t)\big) = 0. \tag{1}$$

This article focuses on the special case[2]

$$V(\phi) = \frac{m^2}{2}\phi^2 + \text{constant} \tag{2}$$

with $m^2 > 0$. In this case, the scalar field is called **free** because the corresponding particles (article 30983) don't interact with each other. This limits the model's usefulness, but it can still be instructive. This article uses the free scalar model to illustrate a few things:

- In relativistic quantum field theory, the vacuum state is entangled with respect to location. Section 10 uses the free scalar model to make this explicit.
- In contrast to the relativistic case, the vacuum state in strictly nonrelativistic quantum field theory is not entangled with respect to location, but that's only because the nonrelativistic approximation implicitly uses a different definition of *location*. Section 14 uses the free scalar model to make this clear.
- In relativistic quantum field theory, the method called **perturbation theory** – expanding in powers of a small parameter in the hamiltonian – doesn't work for state-vectors.[3] Section 15 uses the free scalar model to explain why.
- The concept of *vacuum energy* is meaningless... or not. See section 16.

[1] Each overhead dot denotes a derivative with respect to the time coordinate t, and ∇ is a lattice version of the gradient with respect to the spatial coordinates $\mathbf{x}$. The function V' is the derivative of V with respect to its argument. This article uses natural units, with $\hbar = c = 1$.

[2] Article 44563 showed how to construct this model directly in continuous space. This article uses the lattice version instead, so the field operators may be localized at individual points without causing any mathematical trouble.

[3] Thankfully, it still works for other things.

2 Notation for the spatial lattice

In this article, time is continuous, but space is treated as a finite lattice. The number of dimensions of space is denoted D. Along each of the D axes, the lattice wraps back on itself after K steps,

$$f(\mathbf{x} + K\mathbf{e}_n) = f(\mathbf{x}) \qquad \text{for all } n \in \{1, 2, ..., D\},$$

so that the number K^D of lattice sites is finite. The length of the lattice along each axis will be denoted $L \equiv K\epsilon$. The lattice version of an integral is

$$\int d^D x \; f(\mathbf{x}) \equiv \epsilon^D \sum_{\mathbf{x}} f(\mathbf{x}), \tag{3}$$

where the length scale ϵ is much finer than the resolution of any practical measurements. The lattice version of the Dirac delta distribution for the difference between two points in space is[4]

$$\delta(\mathbf{x} - \mathbf{y}) \equiv \begin{cases} 1/\epsilon^D & \text{if } \mathbf{x} = \mathbf{y}, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The lattice version of the gradient is defined so that

$$\big(\nabla f(\mathbf{x})\big)^2 \equiv \sum_n \left(\frac{f(\mathbf{x} + \mathbf{e}_n) - f(\mathbf{x})}{\epsilon} \right)^2, \tag{5}$$

with one basis vector $\mathbf{e}_n$ of magnitude ϵ for each $n \in \{1, 2, ..., D\}$, and the lattice version of the laplacian is defined by

$$\nabla^2 f(\mathbf{x}) \equiv \sum_n \frac{f(\mathbf{x} + \mathbf{e}_n) + f(\mathbf{x} - \mathbf{e}_n) - 2f(\mathbf{x})}{\epsilon^2} \tag{6}$$

so that integration-by-parts works nicely (article 71852).

[4] For the difference between two momenta, the normalization is different (section 3).

3 Lattice Fourier transforms

The Fourier transform of a function $f(\mathbf{x})$ is defined by (article 71852)[5,6]

$$f(\mathbf{p}) \equiv \int d^D x \; e^{-i\mathbf{p}\cdot\mathbf{x}} f(\mathbf{x}). \tag{7}$$

The allowed values of $\mathbf{p}$ are such that the quantity $\mathbf{p}\cdot\mathbf{x}$ is always an integer multiple of $2\pi/K$. The integral over all $\mathbf{p}$ is defined by

$$\int \frac{d^D p}{(2\pi)^D} \; \cdots \equiv \frac{1}{L^D} \sum_{\mathbf{p}} \cdots . \tag{8}$$

The identities

$$\begin{aligned} \int \frac{d^D p}{(2\pi)^D} \; e^{i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})} &= \delta(\mathbf{x}'-\mathbf{x}) \\ \int d^D x \; e^{i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}} &= (2\pi)^D \delta(\mathbf{p}'-\mathbf{p}) \end{aligned} \tag{9}$$

hold, with $\delta(\mathbf{x}'-\mathbf{x})$ defined by (4) and $\delta(\mathbf{p}'-\mathbf{p})$ defined by

$$(2\pi)^D \delta(\mathbf{p}'-\mathbf{p}) \equiv \begin{cases} L^D & \text{if } \mathbf{p}=\mathbf{p}' \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

The Fourier transforms of $\nabla f(\mathbf{x})$ and $\nabla^2 f(\mathbf{x})$ are $\nabla(\mathbf{p}) f(\mathbf{p})$ and $-|\nabla(\mathbf{p})|^2 f(\mathbf{p})$ with

$$\nabla_k(\mathbf{p}) \equiv \frac{\exp\Big(i\mathbf{p}\cdot\mathbf{e}_k\Big) - 1}{\epsilon} \qquad\qquad |\nabla(\mathbf{p})|^2 = \sum_k \left(\frac{2\sin(\mathbf{e}_k\cdot\mathbf{p}/2)}{\epsilon}\right)^2 . \tag{11}$$

[5] I'm using using continuum-like notation as an abbreviation for lattice expressions, so the integral in (7) is *defined* by (3). I'll occasionally use the explicit lattice notation as a reminder.

[6] I'm using the same letter for the original function and its Fourier transform. They're distinguished from each other by the letters used for their arguments.

4 The model

In the special case (2), the equation of motion (1) reduces to the **Klein-Gordon equation**

$$\ddot{\phi}(\mathbf{x},t) - \nabla^2\phi(\mathbf{x},t) + m^2\phi(\mathbf{x},t) = 0. \tag{12}$$

This is the equation of motion for the free scalar model. It defines the field's time-dependence. The field's algebraic properties are defined by the **equal-time commutation relations** (article 52890)[7]

$$\begin{gathered}\big[\phi(\mathbf{x},t),\phi(\mathbf{y},t)\big] = 0 \qquad\qquad \big[\dot{\phi}(\mathbf{x},t),\dot{\phi}(\mathbf{y},t)\big] = 0 \\ \big[\phi(\mathbf{x},t),\dot{\phi}(\mathbf{y},t)\big] = i\delta(\mathbf{x}-\mathbf{y}).\end{gathered} \tag{13}$$

The field operators are self-adjoint:[8]

$$\big(\phi(\mathbf{x},t)\big)^\dagger = \phi(\mathbf{x},t).$$

Observables are expressed in terms of the field operators, as in article 52890. The field operator $\phi(\mathbf{x},t)$ itself represents an observable localized at the point $\mathbf{x}$ at time t. This defines what *location* means in this model. Section 14 derives an interesting implication of this statement.

[7] $[A,B] \equiv AB - BA$.

[8] In this article, the adjoint of an operator A is denoted $A^\dagger$.

5 The field operators in terms of ladder operators

Define

$$\omega(\mathbf{p}) \equiv \sqrt{m^2 + |\nabla(\mathbf{p})|^2} \tag{14}$$

with $\nabla(\mathbf{p})$ defined by equation (11). This reduces to $\omega(\mathbf{p}) \approx \sqrt{m^2 + \mathbf{p}^2}$ when the components of $\mathbf{p}$ are all close to zero. To simplify the notation, the function $\omega(\mathbf{p})$ will be abbreviated as ω whenever it appears in an expression involving only one wavenumber $\mathbf{p}$.

The equations in section 4 are equivalent to the statement that the field can be written[9,10]

$$\boxed{\phi(\mathbf{x}, t) = \int \frac{d^D p}{(2\pi)^D} \, \frac{a(\mathbf{p}) e^{-i\omega t + i\mathbf{p}\cdot\mathbf{x}} + a^\dagger(\mathbf{p}) e^{i\omega t - i\mathbf{p}\cdot\mathbf{x}}}{\sqrt{2\omega}}} \tag{15}$$

with

$$a^\dagger(\mathbf{p}) \equiv \big(a(\mathbf{p})\big)^\dagger,$$

where the operators $a(\mathbf{p})$ satisfy[11]

$$[a(\mathbf{p}), a(\mathbf{p}')] = 0 \qquad\qquad [a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^D \, \delta(\mathbf{p}' - \mathbf{p}). \tag{16}$$

The operators $a(\mathbf{p})$ and $a^\dagger(\mathbf{p})$ are often called **creation/annihilation operators**.[12] They are also sometimes called **ladder operators**.[13]

[9] The integral in (15) is defined by (8).

[10] The signs of the $\mathbf{p}\cdot\mathbf{x}$ terms in the exponent are chosen so that the sign of $\mathbf{p}$ agrees with the sign of a particle's velocity (article 30983). If we use the mostly-minus convention for the Minkowski metric, then the conditions $p^a = (\omega, \mathbf{p})$ and $x^a = (t, \mathbf{x})$ imply $p_a = (\omega, -\mathbf{p})$ and $\sum_a p_a x^a = \omega t - \mathbf{p}\cdot\mathbf{x}$.

[11] To prove that these equations imply those in section 4, substitute (15) into those equations and then use equations (14) and (16) to simplify the results. Section 6 outlines a proof of the converse.

[12] This name comes from their relationship to particles (article 30983), but beware the name is often applied to *any* set of operators satisfying commutation relations like (16), whether or not they have anything to do with particles.

[13] This name comes from equation (28), which implies that applying $a^\dagger(\mathbf{p})$ to a state increases its energy by $\omega(\mathbf{p})$, and applying $a(\mathbf{p})$ decreases it by the same amount, like climbing up and down the steps of a ladder. Beware the name is sometimes applied to *any* set of operators satisfying commutation relations like (16), whether or not they have anything to do with energy.

6 Proving the converse: outline

To prove that the equations in section 4 imply those in section 5, start by taking the Fourier transform of the equation of motion (12) to get

$$\ddot{\phi}(\mathbf{p},t) + \omega^2 \phi(\mathbf{p},t) = 0, \tag{17}$$

with $\omega(\mathbf{p})$ defined as before and

$$\phi(\mathbf{p},t) \equiv \int d^D x \; e^{-i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{x},t). \tag{18}$$

The fact that $\phi(\mathbf{x},t)$ is self-adjoint implies

$$\big(\phi(\mathbf{p},t)\big)^\dagger = \phi(-\mathbf{p},t). \tag{19}$$

If $\phi(\mathbf{p},t)$ were an ordinary function (instead of an operator), then every solution of equation (17) would be uniquely determined for all t by its initial data $\phi(\mathbf{p},0)$ and $\dot{\phi}(\mathbf{p},0)$. The same should[14] be true when they are operators, so equations (17) and (19) imply

$$\phi(\mathbf{p},t) = \frac{e^{-i\omega t} a(\mathbf{p}) + e^{i\omega t} a^\dagger(-\mathbf{p})}{\sqrt{2\omega}} \tag{20}$$

for some operator $a(\mathbf{p})$. Equation (20) implies

$$a(\mathbf{p}) = \frac{\omega\phi(\mathbf{p},0) + i\dot{\phi}(\mathbf{p},0)}{\sqrt{2\omega}}, \tag{21}$$

and the commutation relations (13) imply that these operators satisfy equations (16). Equation (18) implies

$$\phi(\mathbf{x},t) = \int \frac{d^D p}{(2\pi)^D} \; e^{i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{p},t), \tag{22}$$

and substituting (20) into this equation gives equation (15).

[14] This step in the proof is omitted (hence the word *outline* in the title). We could apply the same reasoning to equation (12) to infer equation (15) directly, but starting with equation (17) makes the omitted step smaller, because the structure of equation (17) is simpler than that of equation (12).

7 The hamiltonian

Let κ denote the constant term in equation (2). When applied to the special case (2), article 52890 shows that the operator

$$H = \int d^D x \left(\frac{\dot{\phi}^2(\mathbf{x},t) + \big(\nabla\phi(\mathbf{x},t)\big)^2 + m^2\phi^2(\mathbf{x},t)}{2} + \kappa \right) \tag{23}$$

is independent of t and that it generates the time dependence of the field operators:

$$\phi(\mathbf{x},t) = U^{-1}(t)\phi(\mathbf{x},0)U(t) \qquad\qquad U(t) \equiv e^{-iHt}. \tag{24}$$

In other words, the operator (23) is the hamiltonian for the free scalar model. The hamiltonian is the observable corresponding to the system's total energy. The state with the lowest possible energy is called the vacuum state.

To derive an explicit expression for that state, start by substituting (22) into (15) and using the identity (9) to get

$$H = \int \frac{d^D p}{(2\pi)^D} \, \frac{\dot{\phi}^\dagger(\mathbf{p},t)\dot{\phi}(\mathbf{p},t) + \omega^2\phi^\dagger(\mathbf{p},t)\phi(\mathbf{p},t)}{2} + L^D\kappa.$$

Now substitute (20) into this. All terms of the form $a(\mathbf{p})a(-\mathbf{p})$ cancel, and so do all terms of the form $a^\dagger(-\mathbf{p})a^\dagger(\mathbf{p})$. In the remaining terms, the t-dependent factors cancel each other, leaving

$$\begin{aligned} H &= \int \frac{d^D p}{(2\pi)^D} \, \frac{\omega}{2}\big(a^\dagger(\mathbf{p})a(\mathbf{p}) + a(-\mathbf{p})a^\dagger(-\mathbf{p})\big) + L^D\kappa \\ &= \int \frac{d^D p}{(2\pi)^D} \, \frac{\omega}{2}\big(a^\dagger(\mathbf{p})a(\mathbf{p}) + a(\mathbf{p})a^\dagger(\mathbf{p})\big) + L^D\kappa. \end{aligned}$$

Use equation (16) to get the final expression

$$H = \int \frac{d^D p}{(2\pi)^D} \, \omega(\mathbf{p})\, a^\dagger(\mathbf{p})a(\mathbf{p}) + \sum_{\mathbf{p}} \frac{\omega(\mathbf{p})}{2} + L^D\kappa. \tag{25}$$

More explicitly,

$$H = \frac{1}{L^D} \sum_{\mathbf{p}} \omega(\mathbf{p})\, a^\dagger(\mathbf{p}) a(\mathbf{p}) + \sum_{\mathbf{p}} \frac{\omega(\mathbf{p})}{2} + L^D \kappa. \tag{26}$$

The constant κ is arbitrary, so we might as well choose it to be[15]

$$\kappa = -\frac{1}{L^D} \sum_{\mathbf{p}} \frac{\omega(\mathbf{p})}{2}. \tag{27}$$

Then

$$\boxed{H = \int \frac{d^D p}{(2\pi)^D}\, \omega(\mathbf{p})\, a^\dagger(\mathbf{p}) a(\mathbf{p}).} \tag{28}$$

As a consistency check, we can use this expression for H and the commutation relations (16) to confirm

$$i\big[H, \phi(\mathbf{x}, t)\big] = \dot{\phi}(\mathbf{x}, t), \tag{29}$$

which is another way of expressing the fact that H generates translations in time.[16]

[15] If we didn't allow κ to depend on the parameters of the lattice, then the hamiltonian's lower bound would become infinite in the infinite-volume limit, because then the sum $\sum_{\mathbf{p}} \omega(\mathbf{p})$ would have an infinite number of terms. That would violate the spectrum condition (section 8).

[16] Equation (29) is the derivative of equation (24) with respect to t.

8 The vacuum state

Each term in the sum (28)[17] is a positive operator, so the spectrum of H has a lower bound, as required by the **spectrum condition** (article 21916). The minimum possible value of $\langle v|H|v\rangle/\langle v|v\rangle$ among all nonzero state-vectors $|v\rangle$ occurs when $|v\rangle$ is the state $|0\rangle$ that satisfies $\langle 0|a^\dagger(\mathbf{p})a(\mathbf{p})|0\rangle = 0$ for all $\mathbf{p}$. This condition is equivalent to[18]

$$a(\mathbf{p})|0\rangle = 0 \qquad\qquad \text{for all } \mathbf{p}, \tag{30}$$

because the norm of a state-vector cannot be zero unless the state-vector itself is zero. Equation (30) is wonderful, because it characterizes the vacuum state algebraically. The vacuum state is defined to be the state with the lowest possible energy, but in most models, we don't know how to characterize the lowest-energy state algebraically. The fact that we can do this in the free scalar model is one of the main reasons this model is so much easier to analyze than most others.

Article 52890 used an explicit representation of the Hilbert space to define the field operators. Thanks to equation (30), we can use that same representation here to derive an explicit expression for the vacuum state.

Here's a quick review of the representation: a state-vector is represented by a complex-valued function $\Psi[s]$ of an enormous number of real variables $s(\mathbf{x})$, one for each point $\mathbf{x}$ in the lattice. This is a **functional representation**.[19] The inner product $\langle 1|2\rangle$ of two state-vectors $\Psi_{|1\rangle}$ and $\Psi_{|2\rangle}$ is the integral of $\Psi^*_{|1\rangle}[s]\Psi_{|2\rangle}[s]$ over all of those variables. At time $t = 0$, the field operator $\phi(\mathbf{x}, 0)$ and its time-derivative $\dot\phi(\mathbf{x}, 0)$ are represented by

$$\phi(\mathbf{x},0)\Psi[s] = s(\mathbf{x})\Psi[s] \qquad\qquad \dot\phi(\mathbf{x},0)\Psi[s] = \frac{-i}{\epsilon^D}\frac{\partial}{\partial s(\mathbf{x})}\Psi[s]. \tag{31}$$

[17] Remember that the "integral" in (28) is defined by (8).

[18] Article 44563 uses a different route to reach this same result.

[19] The word *functional* typically refers to a function whose argument is another function. In this case, the arguments of the function $\Psi[s]$ are real numbers $s(\mathbf{x})$ that can be collectively regarded as a function $s(\mathbf{x})$ from the set of points in space to the set of real numbers.

This is consistent with the commutation relations (13). In this representation, equations (18) and (21) say that the operators $a(\mathbf{p})$ act as[20]

$$a(\mathbf{p})\Psi[s] = \frac{1}{\sqrt{2\omega(\mathbf{p})}} \sum_{\mathbf{x}} e^{-i\mathbf{p}\cdot\mathbf{x}} \left(\epsilon^D \omega(\mathbf{p})\, s(\mathbf{x}) + \frac{\partial}{\partial s(\mathbf{x})} \right) \Psi[s]. \tag{32}$$

Equation (30) is

$$a(\mathbf{p})\Psi_{|0\rangle}[s] = 0 \qquad \text{for all } \mathbf{p}.$$

This has a unique solution (up to proportionality), namely

$$\boxed{\Psi_{|0\rangle}[s] \propto \exp\left(-\epsilon^{2D} \sum_{\mathbf{x},\mathbf{x}'} f(\mathbf{x}'-\mathbf{x}) s(\mathbf{x}) s(\mathbf{x}') \right)} \tag{33}$$

with

$$f(\mathbf{x}'-\mathbf{x}) = \frac{1}{2L^D} \sum_{\mathbf{p}} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\, \omega(\mathbf{p}). \tag{34}$$

The function $\Psi_{|0\rangle}$ defined by (33) is an explicit expression for the vacuum state $|0\rangle$.

Using the notation (3) and (8), the last two equations are[21]

$$\Psi_{|0\rangle}[s] \propto \exp\left(-\int d^D x\, d^D x'\; f(\mathbf{x}'-\mathbf{x}) s(\mathbf{x}) s(\mathbf{x}') \right)$$

$$f(\mathbf{x}'-\mathbf{x}) = \frac{1}{2} \int \frac{d^D p}{(2\pi)^D}\; e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\, \omega(\mathbf{p}).$$

[20] To get the corresponding equation for the adjoint $a^\dagger(\mathbf{p})$, take the complex conjugate and change the sign of the derivative term. The sign-change for the derivative term comes from integrating-by-parts on the right-hand side of the general definition $\int [ds] \Psi_1^*[s] A^\dagger \Psi_2[s] \equiv \int [ds] (A\Psi_1)^*[s] \Psi_2[s]$.

[21] Article 22050 explains how to evaluate the sum (34) in the infinite-volume and continuum limits.

9 Low energy and coarse resolution

Quantum field theory is not usually meant to be a framework for a Theory of Everything. When space is treated as a lattice for the purpose of defining a model, the lattice is understood to be artificial. Such a model is, at best, only intended to agree with experiments at resolutions much coarser than the lattice step-size, so that artifacts due to the discreteness of the lattice are negligible.

To express this mathematically in the free scalar model, let $U(\delta\mathbf{x})$ be a unitary operator that implements a translation with displacement $\delta\mathbf{x}$. The components of $\delta\mathbf{x}$ are integer multiples of ϵ, the lattice step-size. The defining property of $U(\delta\mathbf{x})$ is

$$U^{-1}(\delta\mathbf{x})\phi(\mathbf{x},t)U(\delta\mathbf{x}) = \phi(\mathbf{x}+\delta\mathbf{x},t), \tag{35}$$

because the way the field operators depend on $\mathbf{x}$ is what defines the meaning of *location* in this model. This condition doesn't determine $U(\delta\mathbf{x})$ uniquely, because if $U(\delta\mathbf{x})$ satisfies this condition, then so does $U(\delta\mathbf{x})e^{i\theta(\delta\mathbf{x})}$ for any real-valued function θ. We can make it unique by requiring that the vacuum state be invariant under translations:

$$U(\delta\mathbf{x})|0\rangle = |0\rangle. \tag{36}$$

For predictions involving resolutions much coarser than the lattice spacing, we only need to consider states $|\psi\rangle$ that are not affected much by small translations, which means[22]

$$U(\delta\mathbf{x})|\psi\rangle \approx |\psi\rangle \qquad \text{if } \delta\mathbf{x} \text{ is small.} \tag{37}$$

According to equation (15), the condition (35) is equivalent to the condition

$$U^{-1}(\delta\mathbf{x})a(\mathbf{p})U(\delta\mathbf{x}) = a(\mathbf{p})e^{i\mathbf{p}\cdot\delta\mathbf{x}}. \tag{38}$$

States of the form

$$a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)\cdots|0\rangle \tag{39}$$

[22] If we don't want to rely on the auxiliary condition (36), then we could express the coarse-resolution condition as $|\langle\psi|U(\delta\mathbf{x})|\psi\rangle|^2 \approx |\langle\psi|\psi\rangle|^2$.

span the Hilbert space, and equations (36) and (38) says that the effect of $U(\delta\mathbf{x})$ on such a state is

$$U(\delta\mathbf{x})a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)\cdots|0\rangle = e^{i(\mathbf{p}_1+\mathbf{p}_2+\cdots)\cdot\delta\mathbf{x}}a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)\cdots|0\rangle,$$

so the coarse-resolution condition (37) amounts to the condition that the state $|\psi\rangle$ should be expressible as a superposition of states (39) with each $\mathbf{p}_n$ close to zero – with *close* defined relative to $1/\epsilon$, the inverse of the lattice step-size. In particular, the quantity $\omega(\mathbf{p})$ defined by equation (14) reduces to

$$\omega(\mathbf{p}) \approx \sqrt{m^2+\mathbf{p}^2}$$

at resolutions that are coarse compared to the lattice step-size.

The coarse-resolution condition can be enforced by imposing the low-energy restriction. Equations (16), (28), and (30) imply that the energy of any state of the form (39) is

$$\sum_n \omega(\mathbf{p}_n).$$

If this is small compared to $1/\epsilon$, then the individual $\mathbf{p}_n$s are automatically also small compared to $1/\epsilon$.

This low-energy condition can only be satisfied if the parameter m is also small compared to $1/\epsilon$. That constraint on m is in effect for the rest of this article.

10 Entanglement and the cluster property

A state $\Psi[s]$ is said to be **separable** with respect to locations in space if it's equal to a product of functions, each depending on only one of the variables $s(\mathbf{x})$:[23]

$$\Psi[s] = \prod_{\mathbf{x}} f_{\mathbf{x}}\big(s(\mathbf{x})\big) \qquad \Leftrightarrow \qquad \Psi \text{ is separable w.r.t. location.}$$

Qualitatively, a function that is not separable is called **entangled**. Entanglement (non-separability) can be quantified in many different ways. Use the abbreviation

$$\rho_{|v\rangle}(X) \equiv \frac{\langle v|X|v\rangle}{\langle v|v\rangle}$$

for the expectation value of an operator X in the state $|v\rangle$. The correlation function

$$G_{|v\rangle}(\mathbf{x},\mathbf{y}) \equiv \rho_{|v\rangle}\big(\phi(\mathbf{x},0)\phi(\mathbf{y},0)\big) - \rho_{|v\rangle}\big(\phi(\mathbf{x},0)\big)\rho_{|v\rangle}\big(\phi(\mathbf{y},0)\big)$$

is zero whenever the function $\Psi_{|v\rangle}$ representing $|v\rangle$ is separable with respect to locations in space,[24] so this correlation function is one way to quantify a state's entanglement with respect to location at time $t = 0$. A larger magnitude of this correlation function means more entanglement.

Let's evaluate this correlation function in the vacuum state $|0\rangle$. To do this, start by observing that the expectation value of $\phi(\mathbf{x},0)$ in the vacuum state $|0\rangle$ is zero:[25]

$$\rho_{|0\rangle}\big(\phi(\mathbf{x},0)\big) = 0.$$

[23] More generally, in a model with multiple variables per site, a state is called *separable* with respect to location if it can be written as a product of factors, each of which depends only on the variables associated with one site.

[24] To prove this, use equation (31) for $\phi(\mathbf{x},0)$ and the definition of the inner product described in the text above that equation.

[25] To deduce this, use the fact that the left-hand side must equal its own negative because vacuum state (33) is invariant under $\phi(\mathbf{x},0) \to -\phi(\mathbf{x},0)$.

As a result, the correlation function in the vacuum state reduces to[26]

$$G_{|0\rangle}(\mathbf{x},\mathbf{y}) \propto \big\langle 0\big|\phi(\mathbf{x},0)\phi(\mathbf{y},0)\big|0\big\rangle$$
$$\propto \int [ds]\ \exp\left(-2\epsilon^{2D}\sum_{\mathbf{z},\mathbf{z}'} f(\mathbf{z}'-\mathbf{z})s(\mathbf{z})s(\mathbf{z}')\right) s(\mathbf{x})s(\mathbf{y}),$$

where the integral is over all of the real variables s and the function f is given by (34). To evaluate the integral, think of $s(\mathbf{x})$ as the components of a column matrix with index $\mathbf{x}$, so that the integral may be written

$$G_{|0\rangle}(\mathbf{x},\mathbf{y}) \propto \int [ds]\ e^{-s^T M s/2} s(\mathbf{x})s(\mathbf{y}) \tag{40}$$

where M is the square matrix with components

$$M(\mathbf{x},\mathbf{y}) = 4\epsilon^{2D} f(\mathbf{x}-\mathbf{y}). \tag{41}$$

To evaluate the integral (40), start with

$$F[r] \equiv \int [ds]\ e^{-s^T M s/2 + s^T r}$$
$$= e^{r^T M^{-1} r/2} \int [ds]\ e^{-(s-M^{-1}r)^T M (s-M^{-1}r)} \propto e^{r^T M^{-1} r/2},$$

and then use equation (40) and the definition of $F[r]$ to get

$$\int [ds]\ e^{-s^T M s/2} s(\mathbf{x})s(\mathbf{y}) = \frac{\partial}{\partial r(\mathbf{x})}\frac{\partial}{\partial r(\mathbf{y})} F[r]\bigg|_{r=0} \propto M^{-1}(\mathbf{x},\mathbf{y}).$$

According to equations (34) and (41), this gives

$$G_{|0\rangle}(\mathbf{x},\mathbf{y}) \propto \frac{1}{L^D}\sum_{\mathbf{p}} \frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{\omega(\mathbf{p})} \tag{42}$$

[26]The second step uses the fact that the function f defined in (34) is equal to its complex conjugate.

On a finite lattice, the number of terms in the sum is finite, so (42) is a perfectly well-defined function of $\mathbf{x}-\mathbf{y}$. At resolutions much coarser than the lattice spacing and for distances $|\mathbf{x}-\mathbf{y}|$ much less than the overall size of the lattice, the sum (42) essentially the same as the integral

$$\int \frac{d^D p}{(2\pi)^D} \frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{\sqrt{\mathbf{p}^2+m^2}}.$$

This is demonstrated in article 22050, which also derives the result

$$\int \frac{d^D p}{(2\pi)^D} \frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{\sqrt{\mathbf{p}^2+m^2}} \sim e^{-m|\mathbf{x}-\mathbf{y}|} \tag{43}$$

for large $|\mathbf{x}-\mathbf{y}|$, ignoring an overall nonzero $|\mathbf{x}-\mathbf{y}|$-dependent factor that is practically constant compared to the rapidly decreasing exponential factor. The fact that this is nonzero for $|\mathbf{x}-\mathbf{y}| \neq 0$ implies that the vacuum state is entangled with respect to locations in space.

The result (43) also shows that the entanglement becomes negligible over distances much larger than $1/m$. In particular, this shows that the state $|0\rangle$ has the **cluster property** – a general principle of quantum field theory which says (roughly) that the entanglement of the vacuum state should decrease with increasing distance.[27] The length scale $1/m$ is called the **correlation length**.

[27] On a finite lattice, this principle applies as long as the distance remains much less than the overall size of the lattice. Remember that we are free to take the overall size of the lattice to be much greater than the size of the known universe.

11 The Reeh-Schlieder theorem

Section 10 showed that in the free scalar model, the vacuum state is entangled with respect to location in space. Section 13 will show that all physically sensible states are entangled with respect to location in space. This isn't just a special feature of the free scalar model. In relativistic QFT, every model has this feature.[28] This is a corollary of the **Reeh-Schlieder theorem**. In its most basic form, the Reeh-Schlieder theorem says that for a QFT defined in continuous spacetime, the whole Hilbert space can be generated by acting on the vacuum state with observables in an arbitrarily small neighborhood of a single point in spacetime.[29,30] The proof can be generalized to all physically sensible states.[31,32]

One consequence of the Reeh-Schlieder theorem is that in relativistic QFT, the vacuum state can't be annihilated by any observable that is localized within in a finite region of spacetime.[33] Section 12 shows an example. Witten (2018) explains how this relates to entanglement,[34] and Papadodimas and Raju (2014) and Verch (2005) give additional insight. This also affects the concept of localized particles in relativistic QFT, as explained in article 30983.

[28]Section 1 in Witten (2018) says it like this: "the entanglement [between all field variables with respect to location in space] is not just a property of the states but of the algebras of observables."

[29]More carefully: no state in the Hilbert space is orthogonal to all of these.

[30]Section 4.2.1 in Raju (2020) compares this to an even more interesting result from quantum gravity.

[31]More precisely: it can be generalized to all states that are analytic for the energy operator. Borchers (1965) explains what this means.

[32]Witten (2018) reviews the proof of the basic Reeh-Schlieder theorem in section 2.2 and mentions the generalization in section 2.3: "every state can actually be approximated by states that could be used instead of the vacuum in the Reeh-Schlieder theorem." The proof of the basic version is also reviewed in Araki (1999), theorem 4.14, and the proof of the general version is reviewed in Horuzhy (1990), theorem 1.3.1.

[33]To derive this, let $\Omega(R)$ be the algebra generated by observables in a region R. Suppose that a is an operator for which $a|0\rangle = 0$. Then $\langle\psi|Aa|0\rangle = 0$ for all $A \in \Omega(R)$ and all states $|\psi\rangle$. If a were localized in a region that is spacelike separated from R, then $aA = Aa$ for all $A \in \Omega(R)$, which would imply $\langle\psi|aA|0\rangle = 0$ for all $A \in \Omega(R)$ and all states $|\psi\rangle$. But that contradicts the Reeh-Schlieder theorem, which says that no state is orthogonal to $A|0\rangle$ for all $A \in \Omega(R)$. This shows that a can't be strictly localized in any region that is spacelike separated from R, and since R can be arbitrarily small and arbitrarily far away, a can't be strictly localized in any finite region at all.

[34]Section 2.5 in Witten (2018) clarifies that in continuous spacetime, the entanglement highlighted in this article can't be properly described with respect to a factorization of the Hilbert space, but that subtlety is absent in lattice models, which is the context for the description that was given here in section 10.

12 The Reeh-Schlieder theorem: example

Section 11 mentioned that in relativistic QFT, the lowest-energy state cannot be annihilated by any observable that is strictly localized within in a finite region of space. This section describes an example to reinforce that concept.

By definition, the field operator $\phi(\mathbf{x},t)$ is localized at $\mathbf{x}$ at time t. Equation (11) says that the field operator can be written like this:

$$\phi(\mathbf{x},t) = \phi^+(\mathbf{x},t) + \phi^-(\mathbf{x},t) \tag{44}$$

with

$$\phi^+(\mathbf{x},t) \equiv \int \frac{d^Dp}{(2\pi)^D} \frac{a^\dagger(\mathbf{p})e^{i\omega(\mathbf{p})t - i\mathbf{p}\cdot\mathbf{x}}}{\sqrt{2\omega(\mathbf{p})}} = \big(\phi^-(\mathbf{x},t)\big)^\dagger \tag{45}$$

$$\phi^-(\mathbf{x},t) \equiv \int \frac{d^Dp}{(2\pi)^D} \frac{a(\mathbf{p})e^{-i\omega(\mathbf{p})t + i\mathbf{p}\cdot\mathbf{x}}}{\sqrt{2\omega(\mathbf{p})}}. \tag{46}$$

Equations (16) and (28) imply that these operators decrease and increase (respectively) the energy of any state to which they are applied. In particular, the energy-decreasing part ϕ^- annihilates the vacuum state: $\phi^-(\mathbf{x},t)|0\rangle = 0$.

The field operator $\phi(\mathbf{x},t)$ is local by definition, but it doesn't annihilate the lowest-energy state $|0\rangle$. The energy-decreasing part $\phi^-(\mathbf{x},t)$ does annihilate the lowest-energy state, but the next paragraph shows that it is not local. This is consistent with the Reeh-Schlieder theorem, which implies that a vacuum-annihilating operator like $\phi^-(\mathbf{x},t)$ cannot be localized in any finite region of spacetime, even though the field operator $\phi(\mathbf{x},t)$ itself is localized at $\mathbf{x}$ at time t.

To confirm that $\phi^-(\mathbf{x},t)$ is not localized in any finite region of space at time t, start by using equations (44)-(46) to confirm the identities

$$\int d^Dy\ f(\mathbf{x}-\mathbf{y})\dot{\phi^+}(\mathbf{y},t) = i\phi^+(\mathbf{x},t)$$

$$\int d^Dy\ f(\mathbf{x}-\mathbf{y})\dot{\phi^-}(\mathbf{y},t) = -i\phi^-(\mathbf{x},t)$$

with

$$f(\mathbf{x}-\mathbf{y}) \equiv \int \frac{d^D p}{(2\pi)^D} \, \frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{2\omega(\mathbf{p})}.$$

Add these two identities together, use $\dot{\phi}(\mathbf{x},t) = \dot{\phi^+}(\mathbf{x},t) + \dot{\phi^-}(\mathbf{x},t)$, and re-arrange to get this expression for the energy-decreasing part of the field operator:

$$\phi^-(\mathbf{x},t) = \frac{1}{2}\phi(\mathbf{x},t) + \frac{i}{2}\int d^D\mathbf{y} \; f(\mathbf{x}-\mathbf{y})\dot{\phi}(\mathbf{y},t). \tag{47}$$

On the right-hand side, the first term is local by definition. The last term involves an integral (really a lattice sum) of the local operator $\dot{\phi}(\mathbf{y},t)$ over all points in space, so the last term is not localized in any finite region.

This one example doesn't prove that *every* vacuum-annihilating operator must be nonlocal, but it at least draws attention to the root cause: the function $f(\mathbf{x}-\mathbf{y})$ is nonzero over arbitrarily large distances because of the factor of $\omega(\mathbf{p})$ in the integrand, and the factor of $\omega(\mathbf{p})$ is a consequence designing the model to ensure that Lorentz symmetry emerges at sufficiently low resolution. The usual proofs of the Reeh-Schlieder theorem assume continuous spacetime, but any lattice model that looks like a Lorentz-symmetric quantum field theory at sufficiently low resolution will still have this kind of nonlocal relationship between the field operators (which are local by definition) and their energy-increasing/decreasing parts.

13 Unentangled states have huge energy

Section 10 showed that the vacuum state is entangled with respect to location, and section 11 mentioned that in relativistic quantum field theory, *all* physically sensible states are entangled with respect to location. To corroborate that statement, this section shows that in the free scalar model, any state that is separable (not entangled) with respect to location has huge energy compared to the vacuum state. In the continuum limit $\epsilon \to 0$ with fixed volume L^D, the energy of any separable state becomes infinite.

The hamiltonian is independent of t (article 52890), so we might as well set $t = 0$ and use the abbreviations

$$\phi(\mathbf{x}) \equiv \phi(\mathbf{x}, 0) \qquad\qquad \dot{\phi}(\mathbf{x}) \equiv \dot{\phi}(\mathbf{x}, 0).$$

Start by writing the gradient term in the hamiltonian (23) like this:

$$\begin{aligned} H_\nabla \equiv \epsilon^D \sum_{\mathbf{x}} \frac{\big(\nabla\phi(\mathbf{x})\big)^2}{2} &= \epsilon^D \sum_{\mathbf{x}} \sum_{n} \frac{1}{2} \left(\frac{\phi(\mathbf{x}+\mathbf{e}_n) - \phi(\mathbf{x})}{\epsilon} \right)^2 \\ &= \epsilon^D \sum_{(\mathbf{x},\mathbf{y})} \frac{\phi^2(\mathbf{x}) + \phi^2(\mathbf{y}) - 2\phi(\mathbf{x})\phi(\mathbf{y})}{2\epsilon^2} \end{aligned}$$

where the last sum is over nearest-neighbor pairs $(\mathbf{x}, \mathbf{y})$.[35] Use this to write the hamiltonian (23) as

$$H = \epsilon^D \sum_{\mathbf{x}} h_{\mathbf{x}} + \epsilon^D \sum_{(\mathbf{x},\mathbf{y})} \frac{\phi^2(\mathbf{x}) + \phi^2(\mathbf{y}) - 2\phi(\mathbf{x})\phi(\mathbf{y})}{2\epsilon^2} \tag{48}$$

and

$$h_{\mathbf{x}} \equiv \frac{\dot{\phi}^2(\mathbf{x}) + m^2\phi^2(\mathbf{x})}{2} + \kappa.$$

[35] The cross-terms $\phi(\mathbf{x})\phi(\mathbf{y})$ are sometimes called **hopping** terms, because they allow influences to hop from one site to a neighboring site.

Using the same representation as in section 8, consider a state $\Psi[s]$ that is not entangled with respect to location, which means

$$\Psi[s] = \prod_{\mathbf{x}} \Psi_{\mathbf{x}}\big(s(\mathbf{x})\big)$$

for some collection of single-variable functions $\Psi_{\mathbf{x}}(\cdots)$, one for each point $\mathbf{x}$. Let r denote an individual real variable, and use the abbreviations

$$\langle A \rangle \equiv \frac{\int [ds]\ \Psi^*[s] A \Psi[s]}{\int [ds]\ \Psi^*[s]\Psi[s]} \qquad\qquad \langle A_{\mathbf{x}} \rangle_{\mathbf{x}} \equiv \frac{\int_{-\infty}^{\infty} dr\ \Psi^*_{\mathbf{x}}(r) A_{\mathbf{x}} \Psi_{\mathbf{x}}(r)}{\int_{-\infty}^{\infty} dr\ \Psi^*_{\mathbf{x}}(r)\Psi_{\mathbf{x}}(r)}$$

for the expectation values of any operator A and any single-site operator $A_{\mathbf{x}}$. For any separable state, the expectation value of the hamiltonian (48) is

$$\langle H \rangle = \epsilon^D \sum_{\mathbf{x}} \langle h_{\mathbf{x}} \rangle_{\mathbf{x}} + \epsilon^D \sum_{(\mathbf{x},\mathbf{y})} \frac{\langle \phi^2(\mathbf{x}) \rangle_{\mathbf{x}} + \langle \phi^2(\mathbf{y}) \rangle_{\mathbf{y}} - 2 \langle \phi(\mathbf{x}) \rangle_{\mathbf{x}} \langle \phi(\mathbf{y}) \rangle_{\mathbf{y}}}{2\epsilon^2}. \tag{49}$$

We want to find the minimum possible value of this quantity among all states that are separable with respect to location, so we can compare it to the expectation value of H in the vacuum state. To find the minumum, use the identity

$$\langle A^2 \rangle = \langle (A - \langle A \rangle)^2 \rangle + \langle A \rangle^2$$

and the abbreviations

$$v(A) \equiv \langle (A - \langle A \rangle)^2 \rangle \qquad\qquad v_{\mathbf{x}}(A) \equiv \langle (A - \langle A \rangle_{\mathbf{x}})^2 \rangle_{\mathbf{x}}$$

to write (49) as

$$\begin{aligned} \langle H \rangle = \epsilon^D \sum_{\mathbf{x}} \left(\frac{\big\langle \dot\phi^2(\mathbf{x}) \big\rangle_{\mathbf{x}} + m^2 v_{\mathbf{x}}\big(\phi(\mathbf{x})\big) + m^2 \big\langle \phi(\mathbf{x}) \big\rangle_{\mathbf{x}}^2}{2} + \kappa \right) \\ + \epsilon^D \sum_{(\mathbf{x},\mathbf{y})} \frac{v_{\mathbf{x}}\big(\phi(\mathbf{x})\big) + v_{\mathbf{y}}\big(\phi(\mathbf{y})\big) + \big(\big\langle \phi(\mathbf{x}) \big\rangle_{\mathbf{x}} - \big\langle \phi(\mathbf{y}) \big\rangle_{\mathbf{y}}\big)^2}{2\epsilon^2}. \end{aligned} \tag{50}$$

The value of $\langle\phi(\mathbf{x})\rangle_{\mathbf{x}}$ can be adjusted freely by shifting the function $\Psi_{\mathbf{x}}(r) \to \Psi_{\mathbf{x}}(r + r_0)$, and the $v_{\mathbf{x}}$ terms are invariant under this shift, so equation (50) shows that the separable state that minimizes $\langle H\rangle$ must be such that $\langle\phi(\mathbf{x})\rangle_{\mathbf{x}} = 0$ for every $\mathbf{x}$. For such a state, the quantity (50) reduces to

$$\begin{aligned}\langle H\rangle &= \epsilon^D \sum_{\mathbf{x}} \left(\frac{\big\langle \dot\phi^2(\mathbf{x})\big\rangle_{\mathbf{x}} + m^2 v_{\mathbf{x}}\big(\phi(\mathbf{x})\big)}{2} + \kappa \right) \\ &\quad + \epsilon^D \sum_{(\mathbf{x},\mathbf{y})} \frac{v_{\mathbf{x}}\big(\phi(\mathbf{x})\big) + v_{\mathbf{y}}\big(\phi(\mathbf{y})\big)}{2\epsilon^2} \\ &= \epsilon^D \sum_{\mathbf{x}} \langle \hat h_{\mathbf{x}}\rangle_{\mathbf{x}} \end{aligned} \tag{51}$$

with

$$\hat h_{\mathbf{x}} \equiv \frac{\dot\phi^2(\mathbf{x}) + \hat m^2 \phi^2(\mathbf{x})}{2} + \kappa \qquad\qquad \hat m \equiv \sqrt{m^2 + \frac{2D}{\epsilon^2}}.$$

The constant κ should be chosen so that the vacuum state – which is entangled with respect to location – has finite energy. The conclusion won't depend on specific value of this lowest energy, as long as it's finite, so we might as well choose κ as before (equation (27)). Then, using the same method as in section 7 but without the gradient term and with $\hat m$ in place of m, the minimum possible value of (51) turns out to be

$$\sum_{\mathbf{p}} \frac{\hat m - \omega(\mathbf{p})}{2}.$$

As promised, this diverges in the continuum limit $\epsilon \to 0$ with fixed volume L^D. This completes the proof that in this limit, every state which is separable with respect to location has infinite energy compared to the vacuum state. Physically sensible states should have finite energy compared to the vacuum state, so all physically sensible states are entangled with respect to location.

14 The non-relativistic approximation

Article 30983 shows that $\omega(\mathbf{p})$ is the energy of a single particle with momentum $\mathbf{p}$. In particular, the parameter m in the hamiltonian is the mass (rest energy) of a single particle. The nonrelativistic approximation can be used when a particle's momentum is small compared to its mass: $\mathbf{p}^2 \ll m^2$. Turning this around, the nonrelativistic approximation assumes that m is much larger than other scales of interest. Section 10 showed the entanglement of the vacuum state with respect to location is negligible over distances much larger than the correlation length $1/m$, so this entanglement should be negligible in the nonrelativistic approximation.

That intuition is basically correct, but it misses an interesting detail: when we replace a model with its nonrelativistic approximation (that is, with a model that has the nonrelativistic approximation built into it), we change the relationship between observables and regions of space – we slightly change the meaning of *location*!

To see this, use the expansion

$$\omega(\mathbf{p}) = m + \frac{\mathbf{p}^2}{2m} + O(\mathbf{p}^4) \tag{52}$$

in equation (28) to get

$$H = \int \frac{d^D p}{(2\pi)^D} \left(m + \frac{\mathbf{p}^2}{2m} \right) a^\dagger(\mathbf{p}) a(\mathbf{p}) + O(\mathbf{p}^4) \tag{53}$$

Standard treatments of the nonrelativistic approximation point out that equation (53) may also be written

$$H = \int d^D x \; a^\dagger(\mathbf{x}) \left(m - \frac{\nabla^2}{2m} \right) a(\mathbf{x}) + O(\mathbf{p}^4) \tag{54}$$

with

$$a(\mathbf{x}) \equiv \int \frac{d^D p}{(2\pi)^D} \; e^{i\mathbf{p}\cdot\mathbf{x}} a(\mathbf{p}). \tag{55}$$

Equation (16) implies that these new operators satisfy

$$\big[a(\mathbf{x}),\, a(\mathbf{x}')\big] = 0 \qquad\qquad \big[a(\mathbf{x}),\, a^\dagger(\mathbf{x}')\big] = \delta^D(\mathbf{x}' - \mathbf{x}) \tag{56}$$

exactly. As in section 8, we can use a representation in which a state is a function $\Psi[\sigma]$ of lots of real variables $\sigma(\mathbf{x})$, one per point in space, and in which the effect of the operators $a(\mathbf{x})$ and $a^\dagger(\mathbf{x})$ are

$$\begin{aligned} a(\mathbf{x})\Psi[\sigma] &= \frac{1}{\sqrt{2}}\left(\sigma(\mathbf{x}) + \frac{\partial}{\partial\sigma(\mathbf{x})}\right) \\ a^\dagger(\mathbf{x})\Psi[\sigma] &= \frac{1}{\sqrt{2}}\left(\sigma(\mathbf{x}) - \frac{\partial}{\partial\sigma(\mathbf{x})}\right). \end{aligned} \tag{57}$$

Equations (30) and (55) imply $a(\mathbf{x})|0\rangle = 0$, so the vacuum state in this representation is

$$\Psi_{|0\rangle}[\sigma] \propto \exp\left(-\int d^D x\, \frac{\sigma^2(\mathbf{x})}{2}\right) = \prod_{\mathbf{x}} e^{-\epsilon^D \sigma^2(\mathbf{x})/2}. \tag{58}$$

This shows that the vacuum state is *separable* (not entangled) with respect to the variables $\sigma(\mathbf{x})$. Most importantly, this result is exact – we derived it without using the nonrelativistic approximation at all.[36]

How is this possible? Equation (33) is also exact, and it says that the vacuum state is entangled with respect to location. The resolution of this paradox is that even though the derivation of (58) doesn't use the nonrelativistic approximation, the *interpretation* of the operator $a(\mathbf{x})$ and the variable $\sigma(\mathbf{x})$ as being localized at the point $\mathbf{x}$ *does* use the nonrelativistic approximation. Remember: in the relativistic model, an observable (or other operator) is localized in a given region of spacetime if and only if it can be expressed in terms of the field operators $\phi(\mathbf{x}, t)$ with $\mathbf{x}, t$ inside that region of spacetime. Thanks to the factors of $\omega(\mathbf{p})$ in in equation (21), the definition (55) implies that $a(\mathbf{x})$ cannot be written in terms of only $\phi(\mathbf{x}, 0)$ and $\dot\phi(\mathbf{x}, 0)$ at the same point $\mathbf{x}$. The definition of $a(\mathbf{x})$ involves field

[36] Equations (52)-(54) weren't used, except to motivate the definition (55).

operators from *all* points in space at time $t = 0$, so $a(\mathbf{x})$ is not strictly localized in any finite region at all, much less at the single point $\mathbf{x}$.

On the other hand, the dependence of $a(\mathbf{x})$ on the field operators is mostly concentrated near the point $\mathbf{x}$, falling off exponentially with increasing distance and becoming negligible at distances much greater than $1/m$. For this reason, the interpretation of $a(\mathbf{x})$ and $\sigma(\mathbf{x})$ as being localized at $\mathbf{x}$ is a good approximation – as good as the any other aspect of the nonrelativistic approximation.

One important message is that approximations can slip into the picture through interpretations, not just through equations. Another important message is that *entanglement* always refers to a specific organizing feature of the model, such as location. The nonrelavistic approximation doesn't eliminate the entanglement of the vacuum state with respect to the original spatial structure, but it implicitly defines a new (inequivalent) spatial structure with respect to which the vacuum state's entanglement is zero. Just saying that a state is *entangled*, without any context, makes no sense. Any given state is both entangled and not entangled, depending on which organizing feature is referenced.[37]

[37]The same vacuum state can be written in the representation (33), where it's entangled, or the representation (58), where it's separable, or the representation (60), where it's separable again. To relate the last two representations to each other, use the fact that the $\sigma(\mathbf{x})$ in equation (57) is the Fourier transform of the combination $\sqrt{\omega(\mathbf{p})}\, s(\mathbf{p})$ in equation (59). The equivalence of expressions (58) and (60) is then implied by **Parseval's theorem**.

15 The effect of perturbing the mass parameter

Suppose that H and H' are the hamiltonian (28) for two different values of the mass parameter m. This section derives two related results about the limit $L \to \infty$:

- The vacuum states for H and H' become orthogonal to each other.
- The expectation value of H in the vacuum state for H' becomes infinite.

These results are related to the fact that in the infinite-volume limit $L \to \infty$, two models of free scalar fields with different masses cannot be unitarily equivalent to each other,[38] which in turn is a free-field analog of **Haag's theorem**.[39]

The hamiltonian (28) and the commutation relation (16) may be written more explicitly as

$$H = \frac{1}{L^D} \sum_{\mathbf{p}} \omega(\mathbf{p})\, a^\dagger(\mathbf{p}) a(\mathbf{p})$$

$$[a(\mathbf{p}'),\, a^\dagger(\mathbf{p})] = \begin{cases} L^D & \text{if } \mathbf{p}' = \mathbf{p} \\ 0 & \text{otherwise.} \end{cases}$$

We can enforce the commutation relation using the functional representation

$$a(\mathbf{p}) = \frac{1}{\sqrt{2\omega(\mathbf{p})}} \left(\omega(\mathbf{p}) s(\mathbf{p}) + L^D \frac{\partial}{\partial s(\mathbf{p})} \right) \tag{59}$$

$$a^\dagger(\mathbf{p}) = \frac{1}{\sqrt{2\omega(\mathbf{p})}} \left(\omega(\mathbf{p}) s(\mathbf{p}) - L^D \frac{\partial}{\partial s(\mathbf{p})} \right)$$

where $s(\mathbf{p})$ is an independent real variable for each $\mathbf{p}$. Let Ψ_m denote the vacuum state, using a subscript m to indicate the value of the mass parameter. The vacuum

[38] Klaczynski (2016), theorem 17.1, and Reed and Simon (1975), page 233, theorem X.46.

[39] Haag's theorem was proved in the context of an idealized axiomatic framework that might not cover many models of interest, but its conclusion remains essentially unchanged in less idealized frameworks like lattice quantum field theory.

state is defined by the condition $a(\mathbf{p})\Psi_m = 0$, which is satisfied by the functional

$$\Psi_m[s] \propto \exp\left(-\frac{1}{L^D}\sum_{\mathbf{p}} \omega(\mathbf{p})\,\frac{s^2(\mathbf{p})}{2}\right). \tag{60}$$

This depends on the mass parameter m via $\omega(\mathbf{p})$. The inner product of two vacuum states corresponding to different values m and m' of the mass parameter is

$$\begin{aligned}\langle\Psi_m|\Psi_{m'}\rangle &\propto \int [ds]\ \Psi^*_m[s]\Psi_{m'}[s] \\ &\propto \prod_{\mathbf{p}} \int ds\ \exp\left(-\frac{1}{L^D}\big(\omega(\mathbf{p})+\omega'(\mathbf{p})\big)\,\frac{s^2}{2}\right) \\ &\propto \prod_{\mathbf{p}} \frac{1}{\sqrt{\omega(\mathbf{p})+\omega'(\mathbf{p})}}\end{aligned}$$

where ω and ω' involve m and m', respectively. Therefore,

$$\frac{\big|\,\langle\Psi_m|\Psi_{m'}\rangle\,\big|^2}{\langle\Psi_m|\Psi_m\rangle\,\langle\Psi_{m'}|\Psi_{m'}\rangle} = \prod_{\mathbf{p}} \frac{2}{\eta(\mathbf{p})+1/\eta(\mathbf{p})}$$

with

$$\eta(\mathbf{p}) \equiv \sqrt{\frac{\omega'(\mathbf{p})}{\omega(\mathbf{p})}}.$$

If $m' \neq m$, then $\eta \neq 1$, so each factor in the product is less than 1. (Each factor is equal to 1 when $m' = m$.) In the infinite-volume limit $L \to \infty$ with fixed lattice spacing, the number of factors in the product grows without bound, so the product goes to zero in the limit if $m' \neq m$. This is the first result.

To derive the second result, use the preceding expressions to evaluate

$$\frac{\big|\,\langle\Psi_m|H_{m'}|\Psi_m\rangle\,\big|^2}{\langle\Psi_m|\Psi_m\rangle^2}, \tag{61}$$

where $H_{m'}$ is the hamiltonian (28) with the mass parameter m replaced by m'. When $m' = m$, the quantity (61) is zero. When $m' \neq m$, we can still use the m' version of equation (59) together with the m version of equation (60) to get

$$a_{m'}(\mathbf{p})\Psi_m[s] = \frac{\omega'(\mathbf{p}) - \omega(\mathbf{p})}{\sqrt{2\omega'(\mathbf{p})}} s(\mathbf{p})\Psi_m[s],$$

which implies

$$\begin{aligned}\frac{\left|\,\langle\Psi_m|H_{m'}|\Psi_m\rangle\,\right|^2}{\langle\Psi_m|\Psi_m\rangle^2} &= \frac{1}{2L^D}\sum_{\mathbf{p}}\big(\omega'(\mathbf{p}) - \omega(\mathbf{p})\big)^2 \times \frac{\int [ds]\ \Psi_m^*[s] s^2(\mathbf{p})\Psi_m[s]}{\langle\Psi_m|\Psi_m\rangle^2} \\ &= \frac{\pi L^D}{2L^D}\sum_{\mathbf{p}}\frac{\big(\omega'(\mathbf{p}) - \omega(\mathbf{p})\big)^2}{\omega(\mathbf{p})}.\end{aligned}$$

To evaluate this, use the small-perturbation approximation to get $\omega(\mathbf{p})\delta\omega(\mathbf{p}) \approx m\,\delta m$, and use this in the preceding result to get

$$\frac{\left|\,\langle\Psi_m|H_{m'}|\Psi_m\rangle\,\right|^2}{\langle\Psi_m|\Psi_m\rangle^2} \approx \frac{\pi L^D}{2L^D}\sum_{\mathbf{p}}\frac{m^2(\delta m)^2}{\omega^3(\mathbf{p})}. \tag{62}$$

Depending on the value of D, the quantity

$$\lim_{L\to\infty}\frac{1}{L^D}\sum_{\mathbf{p}}\frac{1}{\omega^3(\mathbf{p})} = \int\frac{d^Dp}{(2\pi)^D}\,\frac{1}{\omega^3(\mathbf{p})}$$

is either infinite or finite, but even if it's finite, the extra factor of L^D make the quantity (62) become infinite in the limit $L \to \infty$. This is the second result.

16 Is vacuum energy real?

The name *vacuum energy* is provocative, the kind of name that science fiction writers love to recite, but is it real? I'll offer a few comments:

- If the energy operator (hamiltonian) is defined to be the generator of translations in time, then only differences between energies matter, because adding or subtracting a constant term to/from the hamiltonian has no effect in the equation

  $$A(t) = e^{-iHt} A(0) e^{iHt}.$$

  In particular, the absolute energy of the vacuum state is arbitrary. Equation (27) exploits this arbitrariness.[40]

- The **Casimir effect** is an observable attractive force between neutral conducting surfaces.[41] It's often described using the words *vacuum energy*, but that language is based on a computational shortcut, as emphasized in Jaffe (2005). Bekenstein (2013) uses another shortcut to help quantify what the usual shortcut ignores. More importantly, though, the models normally used to derive the effect are models in which only energy *differences* matter.

- Gravity makes absolute energies observable,[42] but exactly how this relates to vacuum energy in quantum field theory is still partly mysterious. The cosmological "constant" (which subsumes quantum field theory's vacuum energy) is an adjustable parameter in conventional models of quantum gravity, but those models are either limited to perturbation theory[43] or are limited to a range of cosmological constants that excludes the observed value,[44] so we don't really understand it yet.

[40] Equation (27) also forshadows a subject called **renormalization**: the relationship between a model's inputs (like κ) and outputs (testable predictions, or in this case the arbitrary total energy of the vacuum state) can depend on artificial cutoffs like the granularity ϵ and size L of the lattice.

[41] Example: Bressi *et al* (2002)

[42] I'm being careless with the word *energy* here, but being more careful wouldn't change the basic message.

[43] Donoghue (1995)

[44] Banks (1998), Van Raamsdonk (2016)

17 Energy raising/lowering operators

The operators defined in equations (45) and (46) increase and decrease (respectively) the energy of any state on which they act. The energy-increasing part depends on time through the factor $e^{i\omega(\mathbf{p})t}$ in the integrand, and the energy-decreasing part depends on time through the factor $e^{-i\omega(\mathbf{p})t}$ in the integrand. These are often described as *negative* and *positive frequency*, respectively.[45] This section shows that this correlation between the energy increasing/decreasing property and the negative/positive frequency property is general, not limited to the free scalar model or to models with linear equations of motion.

Let $A(t) \equiv e^{iHt}A(0)e^{-iHt}$ be a time-dependent observable in the Heisenberg picture. Define its positive- and negative-frequency parts by[46]

$$A^{\pm}(t) \equiv \int ds\ f_{\pm}(t-s)A(s)$$

with

$$f_{\pm}(t) \equiv \int_0^{\infty} \frac{d\omega}{2\pi}\ e^{\pm i\omega t - \varepsilon\omega^2}, \tag{63}$$

where $\varepsilon > 0$ is an arbitrarily small regulator. These satisfy

$$A = A^+ + A^-.$$

The positive-frequency part A^- acts as an energy-decreasing operator.[47] To see this, choose two states $|E\rangle$ and $|E'\rangle$ with sharply defined energies E and E'. (This simplifies the calculation, but we could generalize it to almost-sharply defined energies.) and define

$$a(t) \equiv \langle E'|A^-(t)|E\rangle.$$

[45] The sign convention in this terminology might seem backward, but it's common in this context.

[46] This is well-defined if the operator is bounded and $\varepsilon > 0$ (https://math.stackexchange.com/q/3263689).

[47] I'm using the superscript to indicate what the operator does to the energy. Referring to the energy-decreasing part as the *positive*-frequency part is a standard (but arbitrary) convention.

This is zero if $E' > E$. To prove this, use $i\dot{X} = [X, H]$ to get

$$\begin{aligned} HA^-(t) &= A^-(t)H - i\dot{A}^-(t) \\ &= \int ds \int_0^\infty \frac{d\omega}{2\pi}\, e^{-i\omega(t-s)-\varepsilon\omega^2} A(s)(H-\omega). \end{aligned}$$

Sandwich this between the states $|E\rangle$ and $|E'\rangle$ to get

$$E'a(t) = \int ds \int_0^\infty \frac{d\omega}{2\pi}\, e^{-i\omega(t-s)-\varepsilon\omega^2}(E-\omega)\langle E'|A(s)|E\rangle.$$

Take the Fourier transform of both sides with respect to t to get

$$E'g(\omega) = (E-\omega)g(\omega) \tag{64}$$

with

$$g(\omega) \equiv \theta(\omega \geq 0) \int ds\; e^{i\omega s-\varepsilon\omega^2}\langle E'|A(s)|E\rangle.$$

Since $\omega \geq 0$, this implies $g(\omega) = 0$ if $E' > E$. The quantity $a(t)$ is the Fourier transform of $g(\omega)$, this implies $a(t) = 0$ if $E' > E$, so $A^-(t)$ can only decrease state's energy, if it changes the energy at all.

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