Complete Gauge Fixing

Randy S

Abstract Articles 51376 and 89053 describe ways to construct models involving quantum gauge fields when space or spacetime is treated as a lattice. In those models, the gauge field is described as a collection of G-valued **link variables**, one for each directed link in the lattice, where G is the gauged group. The collection of link variables used in those constructions is overcomplete: the same model (on the same lattice) can be described using only a subset of the link variables. This article explains how to choose a subset that is just barely complete. This is a type of **gauge fixing**. Important special cases of this type of gauge fixing include the **temporal gauge** and the **axial gauge** (after extending them slightly so that only a *minimal* complete set of link variables remains). This type of gauge fixing can be useful for understanding a model's general properties. Gauge fixing is also a prerequisite for defining small-coupling expansions. This article also shows that even after complete gauge fixing, some **zero modes** may remain.

Contents

1 Points and links

This section describes a general framework that includes the space(time) lattices used in articles 51376 and 89053 as special cases.

An **(undirected) graph** $\Gamma = (P, \tilde{L})$ consists of a set P of **points**[1] and a set $\tilde{L}$ of **(undirected) links**[2] Each link $\{x, y\} \in \tilde{L}$ is an unordered pair of points x and y with $x \neq y$.[3] We can think of a **lattice** as a special type of graph, one whose points P are all generated from a single point by adding integer multiples of the basis vectors $\mathbf{e}_k$.[4]

A **directed link** is an ordered pair (x, y) of points. For each undirected link $\{x, y\}$, we can define two corresponding **directed links**, (x, y) and (y, x). The set of all directed links corresponding to the undirected links in $\tilde{L}$ will be denoted L. Both directions are included, so if $(x, y) \in L$, then $(y, x) \in L$. In this article, the word *link* without any qualifiers means *undirected link*.

A graph is called **finite** if the number of points in P is finite. A graph is called **connected** if $\tilde{L}$ cannot be partitioned into two subsets whose links don't share any points with each other. In this article, graphs are always assumed to be finite and connected.

Every point in P should be used in at least one link in $\tilde{L}$. A point in P that belongs to only one link in $\tilde{L}$ will be called a **boundary point**.[5] Other points will be called **interior points**.

[1] This language alludes to typical applications in lattice quantum field theory (like in article 51376) where the points P form a lattice and the links represent nearest-neighbor pairs of points in the lattice.

[2] In the literature about graph theory, a point is usually called a **vertex**, and a link is usually called an **edge**.

[3] Pairs of the form $\{x, x\}$ are not allowed in this article.

[4] The bottom of page 63 in Harlow and Ooguri (2021) clarifies this use of the word *lattice*.

[5] Article 51376

2 Configurations and gauge transformations

Let G be the gauged group (a compact Lie group), and let 1 denote the identity element of G. If u is a map from L to G, then the value (the element of G) that u assigns to a link $\ell = (x, y) \in L$ will be denoted either $u(\ell)$ or $u(x, y)$. A map with the property

$$u(y, x) = \big(u(x, y)\big)^{-1}$$

will be called a **configuration**. A configuration will be called **trivial** on the link $\{x, y\}$ if $\ell(x, y) = 1$.

Let h denote any map from P to G. Given any configuration u with values in G, we can define a new configuration u^h by

$$u^h(x, y) \equiv h(x)u(x, y)\big(h(y)\big)^{-1}. \tag{1}$$

The transformation $u \to u^h$ is called a **gauge transformation**. A gauge transformation will be called **trivial** at the point x if $h(x) = 1$.

In models with gauge fields, observables are required to be invariant under gauge transformations, or at least under gauge transformations that are trivial at all boundary points. This implies that the set of link variables is overcomplete, because gauge transformations may be used to change some of the link variable values to 1 without changing any observables. This article explains how to choose a minimal complete set of link variables and how to construct a gauge transformation that changes the values of all the other link variables to 1.

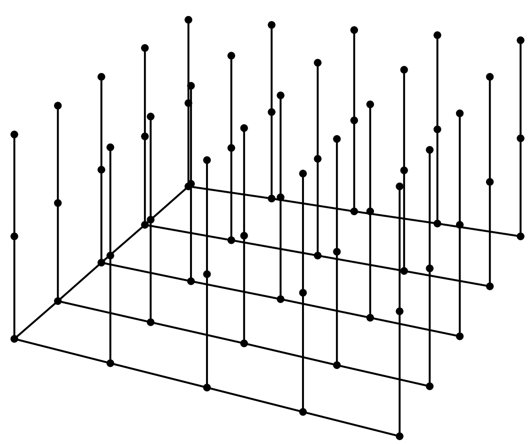
Choosing a gauge (also called **gauge fixing**) means choosing a condition that each configuration of the gauge field can be made to satisfy by applying the right gauge transformation, and then considering only configurations that satisfy that condition. In this article, the condition is that all the link variables outside the given minimal complete set are equal to 1.

3 The concept of a maximal tree

Given a graph $\Gamma = (P, \tilde{L})$, a **tree** is a subset of $\tilde{L}$ that doesn't have any closed loops. Any finite connected undirected graph $\Gamma = (P, \tilde{L})$ has a **maximal tree**,[6] which is a subset $T \subset \tilde{L}$ such that any two points in P are connected to each other by exactly one path (connected series of links) in T. A maximal tree T has these properties:

- It doesn't have any loops, because any two points in P are connected by only one path in T.
- Adding any other link in $\tilde{L}$ to T would create a loop, because that link's endpoints are already connected by a path in T.

This picture shows an example of a maximal tree for the graph $(P, \tilde{L})$, where P is a three-dimensional grid of points of size $3 \times 5 \times 5$ and $\tilde{L}$ consists of all nearest-neighbor pairs of points:



Each solid line segment represents a link in the maximal tree.

[6]Montvay and Münster (1997), end of section 3.2.5

4 Gauge fixing on a maximal tree

Given a graph $\Gamma = (P, \tilde{L})$, a configuration of the gauge field, and a maximal tree $T \subset \tilde{L}$, this section constructs a gauge transformation that makes $u(\ell) = 1$ for every link ℓ in the maximal tree. This would not be possible for any larger subset of $\tilde{L}$, because any larger subset would have at least one loop, and configurations of the gauge field exist for which the product of link variables around a given loop (regardless of the start-point) is not gauge-equivalent to 1.

The promised gauge transformation will be constructed as a composition of gauge transformations that each affects only a single point ($h(x) = 1$ for all but one point x). To begin, choose any two distinct points $x_0, x_N \in P$, and let $x_0, x_1, x_2, ..., x_N$ be the unique sequence of points for which the links (x_{k-1}, x_k) are in T for all $k \in \{1, 2, ..., N\}$. Apply a gauge transformation with $h(x) = 1$ at all points except x_1, and choose $h(x_1)$ so that the gauge transformation changes the value of $u(x_0, x_1)$ to 1. This also changes the value of $u(x_1, x_2)$. Then apply a gauge transformation with $h(x) = 1$ at all points except x_2, and choose $h(x_2)$ to change that new value of $u(x_1, x_2)$ to 1. Continue like this until all the link variables along the given path from x_0 to x_N are equal to 1. Now choose any point in P that is not on the original path, and consider the unique path in T from x_0 to P. Apply the same recipe along that path to make all the link variables along that path equal to 1. Then choose any point that is not in any of the preceding paths, and apply the same recipe again along the unique path in T from x_0 to that new point. Repeat this until all the points in P have participated in at least one of the paths. The result is that all the link variables associated with links in T are equal to 1, as promised.

5 Gauge fixing using only interior transformations

Now suppose that gauge fixing process is only allowed to use **interior gauge transformations**, defined to be gauge transformations for which $h(x) = 1$ at every boundary point x.

If the graph has only one boundary point, then the construction described in section 4 still works if we take x_0 to be that one boundary point, because that construction only uses gauge transformations with $h(x_0) = 1$.

If the graph has more than one boundary point, then we cannot set all the link variables on a maximal tree equal to 1 using only interior gauge transformations, but we can do this on a slightly smaller tree. Let $\Gamma_I = (P_I, \tilde{L}_I)$ be the graph obtained from Γ by omitting all boundary points and the links to which they belong,[7] and let T_I be a maximal tree for Γ_I. Choose any point $x_1 \in P_I$ that is paired with a boundary point by one of the links in $\tilde{L}$, and let x_0 denote that boundary point. Define T to consist of the links in T_I together with the link $\{x_0, x_1\}$. Then the construction described in section 4 may be used to make all the link variables associated with links in T equal to 1. The set T is not contained in any larger set with that property, because any larger set either has a loop or has more than one boundary point.

[7]Mnemonic: the subscript I stands for *interior points only*.

6 The axial and temporal gauges

In applications to quantum field theory, the graph Γ is usually a (hyper)cubic lattice, either truncated or wrapped so that the total number of points is finite. In that case, every link is parallel to one of the dimensions of the lattice. We can choose a tree T that includes all or at least almost all of the links parallel to a selected dimension.[8] Choosing a tree that includes all the links parallel to that direction might not be possible, because that might produce a loop (if the lattice wraps back on itself) or because it might include more than one boundary point (which is a problem if the gauge fixing process is only allowed to use interior gauge transformations).[9] Ignoring that technicality, making the link variables equal to 1 for all the links in T gives what is called an **axial gauge**,[10] or a **temporal gauge** if the chosen dimension is timelike.

[8]The picture in section 3 illustrates this for the vertical direction.

[9]Section 5

[10]The root word "axis" refers to the chosen dimension.

7 Gauge fixed integrals

Consider an integral of the form

$$\omega \equiv \int [du]\; F[u] \tag{2}$$

where $F[u]$ is a function of the link variables and the integral is over all the link variables, defined using the Haar measure. This section introduces another way of writing the integral that gives the same answer whenever the function $F[u]$ is invariant under gauge transformations.[11]

Choose a set undirected links that form a maximal tree. Let T be the set of directed links corresponding to the undirected links in the given maximal tree, and let $u(T)$ denote the associated set of link variables. Consider the obvious identity

$$\omega = \int \left(\prod_{\ell\in T} du(\ell)\right) f\big[u(T)\big] \qquad\qquad f\big[u(T)\big] \equiv \int \left(\prod_{\ell\notin T} du(\ell)\right)\; F[u].$$

If $F[u]$ is gauge invariant, then the quantity $f\big[u(T)\big]$ is also gauge invariant and depends only on the link variables in $u(T)$, so the result derived in section 4 implies that f is actually a constant: it doesn't depend on the values of the link variables in $u(T)$. This gives

$$\omega \propto \int \left(\prod_{\ell\notin T} du(\ell)\right)\; F[u'] \qquad\qquad u'(\ell) = \begin{cases} u(\ell) & \text{if } \ell\notin T \\ 1 & \text{if } \ell\in T. \end{cases}$$

[11] The result derived in this section can be adapted to cases where $F[u]$ is invariant only under interior gauge transformations, by using a tree like the one defined in section 5 instead of using a maximal tree.

8 Gauge fixing and the path integral

Consider a model whose only field is the gauge field represented by the link variables $u(\ell)$, and suppose the action has the standard form

$$S[u] = \beta \sum_{\square} \left(1 - \frac{W(\square)}{N}\right) \tag{3}$$

with $\beta > 0$, the sum is over all unoriented plaquettes, N is the trace of the identity matrix, and $W(\square)$ is a **traced plaquette variable** – the trace of the product of link variables (in a faithful matrix representation) around the perimeter of an oriented version of $\square$.[12] The integrand of the euclidean path integral includes the factor $\exp(-S[u])$, so when β is large, this factor suppresses configurations the contribution of any configuration in which any of the quantities $W(\square)/N$ deviates significantly from 1. To implement a small-coupling expansion, though, we need a stronger kind of suppression: we want to suppress all configurations whose individual link variables deviate significantly from 1.

The gauge fixing protocol described in the previous sections can help achieve that goal. To avoid technical complications, consider a lattice that doesn't wrap back on itself and doesn't have any boundary points.[13] Choose a maximal tree T that includes all the links parallel to the first dimension, all the links parallel to the second dimension whose first coordinate is zero, all the links parallel to the third dimension whose first two coordinates are zero, and so on. Section 3 depicts an example of such a tree for a three-dimensional lattice. Every configuration of the gauge field may be converted by a gauge transformation to a configuration that has all link variables associated with links in T equal to 1, so the integrals over those link variables may be omitted without affecting any of the model's predictions.[14] With those link variables permanently constrained to be equal to 1,

[12] Without the trace, the product (ordered around the perimeter starting from a given basepoint) is called a **plaquette variable** (Montvay and Münster (1997), text after equation (3.63)).

[13] If the lattice wraps back on itself, then some unsuppressed **zero modes** may remain even after completely fixing the gauge (section 9).

[14] In practice, most calculations use a different type of gauge fixing that respects the symmetries of spacetime.

the factor $\exp(-S[u])$ suppresses all configurations in which any of the remaining link variables differs significantly from 1. To deduce this, refer to the example depicted in section 3 and use these fact:

- Each of the remaining link variables belongs to at least one plaquette that involves no more than one other remaining link variable.

- Each of those plaquettes is part of a "ladder" of plaquettes that starts with a plaquette involving only one of the remaining link variables. A "ladder" is a sequence of plaquettes such that any two consecutive plaquettes in the sequence share one of the remaining link variables with each other.

- If a plaquette involves only one of the remaining link variables, then $W(\square)/N$ is close to 1 only if that link variable is close to 1.

- If a plaquette involves two of the remaining link variables, then $W(\square)/N$ is close to 1 only if those link variables are close to being inverses of each other, so if one of them is close to 1, then so is the other one.

To finish the argument, we need to show that small differences between the values of nearby link variables cannot accumulate into large differences between link variables on opposite sides of the lattice. That follows from the fact that the factor $\exp(-S[u])$ is a product of factors, one for each plaquette, so the accumulation of small differences over a long sequence of plaquettes is suppressed by the product of the same number of already-small factors.[15]

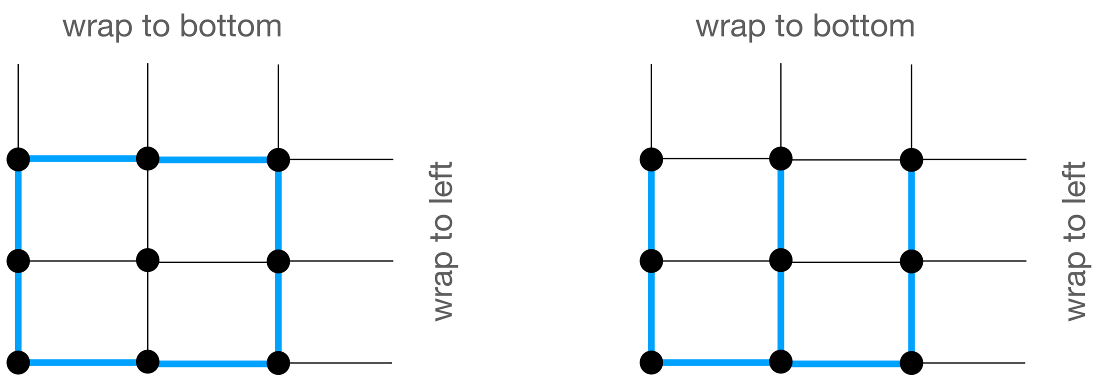
Instead of setting individual link variables to 1, those schemes work by constraining certain combinations of link variables. Those constraints are enforced with the help of additional fields called **ghost fields** (Montvay and Münster (1997), pages 122-123). Preserving spacetime symmetries is convenient for many calculations, but the type of gauge fixing described in this article is conceptually simpler because it doesn't require introducing ghost fields (Montvay and Münster (1997), text below equation (3.133)).

[15] This argument can be made quantitative, but I won't bother doing that here.

9 Zero modes for an abelian gauged group

Section 8 mentioned that one of the motives for using gauge fixing is to enable a small-coupling expansion and showed that gauge fixing achieves that goal for at least some lattices. This section shows that gauge fixing does not always achieve that goal: if the lattice is wrapped and $G = U(1)$,[16] then the action is still independent of some directions in the space parameterized by the remaining field variables even after fixing the gauge on a maximal tree. Those directions are called **zero modes**. The presence of zero modes prevents using a small-coupling expansion unless something is done to remove them.

Consider a d-dimensional hypercubic lattice that is wrapped in each dimension – the lattice version of a d-dimensional torus – with K sites along each dimension. Consider the largest possible d-dimensional hypercube C that doesn't intersect itself as a result of the lattice's wrapping. We can choose a maximal tree that doesn't involve any links outside of C, using a pattern like the example depicted in section 3. An example with $d = 2$ and $K = 3$ is depicted here:



The picture on the left highlights the outline of C, and the picture on the right highlights the links in a maximal tree. The set of links outside of C is the union of sets $\tilde{C}_1, ..., \tilde{C}_d$, where $\tilde{C}_k$ is the set of K^{d-1} links that are both outside C and parallel

[16]For $G = U(1)$, the link variables all commute with each other, and a plaquette variable is the same as an untraced plaquette variable – just a product of link variables around the perimeter of a plaquette.

to the kth dimension. (Each of the links in $\tilde{C}_k$ connects one of C's hyperfaces to the opposite hyperface by wrapping around the kth dimension.) For each $k \in \{1, ..., d\}$, multiplying all the link variables in the set $\{u(\ell) \mid \ell \in \tilde{C}_k\}$ (using a single consistent orientation for all those links) by the same element of the gauged group leaves all plaquette variables invariant. This multiplication is a translation in the space parameterized by the link variables. This is a zero mode, because the action (3) depends only on the plaquette variables. This shows that at least d zero modes exist after gauge fixing on this particular maximal tree. The same conclusion must be true for any maximal tree, because every maximal tree has the same number of links, and the choice of maximal tree does not affect the number of independent plaquette variables.

To check that conclusion, we can use a different approach to count the number of zero modes when $G = U(1)$ and $d = 3$. For any d, if the lattice has K sites along each dimension, then the total number of (unoriented) links is $d \cdot K^d$, and the number of links in any maximal tree is $K^d - 1$ (the number of sites minus one), because that's just enough links to include a unique path between any two sites. This implies that the number of independent link variables after gauge fixing on a maximal tree is

$$d \cdot K^d - (K^d - 1) = (d - 1)K^d + 1. \tag{4}$$

From here, we can determine the number of zero modes by subtracting the number of independent plaquette variables, because those are the combinations of link variables on which the action (3) depends. For $d = 3$, the number of (unoriented) plaquettes is $3K^3$, but the corresponding plaquette variables are not all independent, because the product of a set of plaquette variables is 1 whenever those plaquettes form a closed surface (with consistent orientations). The number of independent closed surfaces is $K^3 - 1 + 3$, where K^3 is the number of cubettes.[17] The 1 is subtracted because any one cubette variable is proportional to the sum of the others (because the lattice wraps in every dimension), and the 3 is added to account for the existence of one surface that is wrapped in two dimensions, for each of the 3 pairs of dimensions. This shows that the number of independent plaquette

[17] The boundary of every cubette a closed surface.

variables is

$$3K^3 - (K^3 + 2) = 2K^3 - 2. \tag{5}$$

The difference between (4) (with $d = 3$) and (5) is 3, so any function of the plaquette variables – like the action (3) – must be independent of at least 3 directions in the space parameterized by the link variables even after gauge fixing on a maximal tree. In other words, this model has 3 zero modes, which agrees with the result in the previous paragraph.

10 References

(Open-access items include links.)

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11 References in this series

Article **51376** (https://cphysics.org/article/51376):
"The Quantum Electromagnetic Field on a Spatial Lattice"

Article **89053** (https://cphysics.org/article/89053):
"Quantum Gauge Fields on a Spacetime Lattice"