Units in Electrodynamics

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Abstract **Natural units** are convenient for abstract theoretical considerations, because they help keep the equations clean. For practical applications, though, other systems of units are normally used. Two of the most commonly used systems are **International System of units (SI)** and **Gaussian units**. Gaussian units can be regarded as a compromise between natural units and SI units, but Gaussian units introduce a new artifact that is absent in both of the other systems. **Heaviside-Lorentz units** is a compromise similar to Gaussian units, but without the artifact. This article introduces these systems, with special emphasis on the International System.

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1 Natural units: equations of motion

Article 31738 introduced Maxwell's equations. When specialized for 3-dimensional space, they look like this:

$$\begin{aligned} \nabla\cdot\mathbf{E} &= \rho & \qquad \nabla\times\mathbf{B} &= \frac{\partial\mathbf{E}}{\partial t}+\mathbf{J} \\ \nabla\cdot\mathbf{B} &= 0 & \qquad \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t} \end{aligned} \tag{1}$$

and the Lorentz force equation (article 54711) looks like this:

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B}). \tag{2}$$

In these equations, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, ρ is the charge density, and $\mathbf{J}$ is the current density, $\mathbf{p}$ is the particle's momentum, and q is its charge.

Equations (1)-(2) are written in **natural units**. In natural units, any two quantities that can be mixed with each other by spacetime coordinate transformations (article 31738) are both expressed in the *same* units:

- The space coordinates and the time coordinate are both expressed in the same units.
- Energy and momentum are both expressed in the same units.
- Charge density and current density are both expressed in the same units.
- The electric and magnetic fields are both expressed in the same units.

2 Natural units: dimensional analysis

Let $[X]$ denote the units in which the quantity X is expressed. We can use this notation to summarize a system of natural units for the quantities in equations (1)-(2). Use L to denote a unit of length, M a unit of mass, and D the number of spatial dimensions.[1] The relationship

$$[\rho] = \frac{[q]}{L^D} \tag{3}$$

enforces the requirement that ρ represents a charge density. Easy consequences of equations (1) and (2) include

$$[\mathbf{v}] = 1 \qquad\qquad [\mathbf{E}] = [\mathbf{B}] \qquad\qquad [\mathbf{J}] = [\rho].$$

Combining (3) with the upper-left equation in (1) gives $[\mathbf{E}/q] = L/L^D$, and combining $[\mathbf{p}] = M$ with equation (2) gives $[q\mathbf{E}] = M/L$. Combine these two results to get

$$[\mathbf{E}] = \sqrt{\frac{M}{L^D}} \qquad\qquad [q] = \sqrt{ML^{D-2}}.$$

Combine the second of these equations with (3) to get

$$[\rho] = \sqrt{\frac{M}{L^{D+2}}}.$$

Altogether, this expresses the units of each quantity in equations (1) and (2) in terms of M and L.

[1] Equations (1) and (2) are written for $D = 3$. The generlization to arbitrary D affects the pattern of indices, but the pattern of indices doesn't affect the analysis of units (also called dimensional analysis).

3 SI units: equations of motion

All of the quantities in equations (1)-(2) may be naturally expressed using units of length and mass. However, as shown in section 2, the derived units involve square roots, which can be inconvenient in practical applications. The International System (SI) of units[2,3] avoids this inconvenience at the cost of introducing two new dimensionful coefficients into the equations. These two new coefficients are denoted ϵ_0 and μ_0 and are called the **permittivity** and **permeability** of free space, respectively.

In the SI system, equations (1)-(2) become[4]

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{B} = \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} + \mu_0\mathbf{J}$$
$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{4}$$

and the Lorentz force equation is still

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}). \tag{5}$$

The coefficients c, ϵ_0, and μ_0 are related to each other by

$$c\sqrt{\epsilon_0\mu_0} = 1. \tag{6}$$

The coefficient c is called the **speed of light** because equations (4) imply that the speed of an electromagnetic wave is equal to c when matter is absent (that is, when $\rho = 0$ and $\mathbf{J} = 0$).

[2] SI Brochure (2019)

[3] The SI system has also been called the **(rationalized) MKSA** system. "MKSA" stands for meters, kilograms, seconds, and amperes.

[4] Simpson (2013)

4 From natural units to SI units

The SI form of the equations can be obtained from equations (1)-(2) by making these replacements:[5]

$$
\begin{gathered}
t \mapsto ct \qquad\qquad \mathbf{v} \mapsto \mathbf{v}/c \\
\mathbf{E} \mapsto \mathbf{E}\sqrt{\epsilon_0} \qquad\qquad \mathbf{B} \mapsto \mathbf{B}\,c\sqrt{\epsilon_0} \\
q \mapsto \frac{q}{\sqrt{\epsilon_0}} \qquad\qquad \rho \mapsto \frac{\rho}{\sqrt{\epsilon_0}} \qquad\qquad \mathbf{J} \mapsto \frac{\mathbf{J}}{c\sqrt{\epsilon_0}}.
\end{gathered}
\tag{7}
$$

[5] Here, the notation $X \to Y$ means "X in natural units is equal to Y in SI units."

5 SI units: dimensional analysis

Start with

$$[\nabla] = \frac{1}{\text{meter}} \qquad \left[\frac{\partial}{\partial t}\right] = \frac{1}{\text{second}}$$

$$\left[\frac{\partial \mathbf{p}}{\partial t}\right] = \text{newton} \qquad [q] = \text{coulomb} \qquad [\rho] = \frac{\text{coulomb}}{\text{meter}^D}$$

along with the constraint

$$c^2\epsilon_0\mu_0 = 1$$

and the definition

$$\text{ampere} \equiv \frac{\text{coulomb}}{\text{second}}.$$

The top row of equations (4) gives

$$[\mathbf{J}] = [c\rho] = \frac{\text{ampere}}{\text{meter}^{D-1}}.$$

Equation (5) gives

$$[\mathbf{E}] = \frac{\text{newton}}{\text{coulomb}} \qquad [\mathbf{B}] = [\mathbf{E}/c] \equiv \text{tesla}.$$

Finally, these imply

$$[\epsilon_0] = \frac{\text{coulomb}^2}{\text{newton} \cdot \text{meter}^{D-1}} \qquad [\mu_0] = \frac{\text{newton} \cdot \text{meter}^{D-3}}{\text{ampere}^2}$$

For any D, all of the square-roots that were encountered in section 2 with natural units are eliminated in SI units.

6 SI units: a mnemonic device for 3d space

One way to remember the units of μ_0 and ϵ_0, and their approximate values, is to remember that speed of light and the **impedance of free space** are

$$c \approx 3 \times 10^8 \text{ m/s} \qquad\qquad Z_0 = \mu_0 c \approx 377\,\text{ohms},$$

together with the relationships

$$c^2\epsilon_0\mu_0 = 1 \qquad\qquad Z_0 = \mu_0 c.$$

Use those relationships to get

$$\epsilon_0 = \frac{1}{Z_0 c} \qquad\qquad \mu_0 = \frac{Z_0}{c}.$$

To relate the units of Z_0 to to the units of $\mathbf{E}$ and $\mathbf{B}$, use[6,7]

$$\text{volt} \equiv \frac{\text{watt}}{\text{ampere}} \qquad\qquad \text{ohm} \equiv \frac{\text{volt}}{\text{ampere}}.$$

[6]SI Brochure (2019)

[7]In words: power is current times voltage, and voltage is current times resistance.

7 Heaviside-Lorentz units: equations of motion

The Heaviside-Lorentz system of units is a compromise between natural units and SI units. In the SI system, distances and time intervals are expressed in different units, and electric and magnetic fields are also expressed in different units. The Heaviside-Lorentz system treats distances and time intervals as in the SI system, but it expresses electric and magnetic fields in the same units (as the natural system does).

In the Heaviside-Lorentz system, equations (1)-(2) become[8]

$$\begin{aligned} \nabla\cdot\mathbf{E} &= \rho \qquad & \nabla\times\mathbf{B} &= \frac{1}{c}\left(\frac{\partial\mathbf{E}}{\partial t}+\mathbf{J}\right) \\ \nabla\cdot\mathbf{B} &= 0 \qquad & \nabla\times\mathbf{E} &= -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \end{aligned} \tag{8}$$

and

$$\frac{d\mathbf{p}}{dt} = q\left(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}\right). \tag{9}$$

[8] Simpson (2013)

8 Gaussian units: equations of motion

In the Gaussian system, equations (1)-(2) become[9]

$$\nabla \cdot \mathbf{E} = 4\pi\rho \qquad \nabla \times \mathbf{B} = \frac{1}{c}\left(\frac{\partial \mathbf{E}}{\partial t} + 4\pi\mathbf{J}\right)$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{10}$$

and

$$\frac{d\mathbf{p}}{dt} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right). \tag{11}$$

These equations differ from the Heaviside-Lorentz versions (8)-(9) only in the factors of 4π that multiply the charge and current densitites. The next section puts this difference in perspective.

[9]Simpson (2013)

9 Gaussian units and the factors of 4π

In 3-dimensional space, the equation $\nabla \cdot \mathbf{E} \propto \rho$ and the Lorentz force equation imply that the electrostatic force between two charges q_1 and q_2 is proportional to $q_1 q_2/r^2$, where r is the distance between them. This is **Coulomb's law**. Historically, Coulomb's law was known before equations (1)-(2) were known, but Maxwell's equations are more fundamental because they govern the behavior of electromagnetic fields and charged particles under *all* conditions,[10] not just the special circumstance represented by Coulomb's law.

The Gaussian system of units eliminates a factor of 4π from Coulomb's law by introducing factors of 4π into Maxwell's equations. This is a classic case of "letting the tail wag the dog."

The standard definition of Newton's gravitational constant is backward in the same way: it *introduces* an unnatural factor of 4π into the general equation (the field equation of general relativity) in order to *eliminate* such a factor from a special solution in 3-dimensional space.[11] Using a convention that complicates general principles in order to simplify a special case eventually causes more trouble than it saves.

[10] This statement ignores quantum physics, but that doesn't affect the message that is being emphasized here.
[11] Robinson (2006)

10 References

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11 References in this series

Article **31738** (https://cphysics.org/article/31738):
"The Electromagnetic Field and Maxwell's Equations in Any Number of Dimensions"
(version 2024-05-21)

Article **54711** (https://cphysics.org/article/54711):
"Charged Particles in an Electromagnetic Field: the Lorentz Force Equation in Flat Spacetime"
(version 2024-05-21)