Diffeomorphisms, Tensor Fields, and General Covariance

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Abstract Tensor fields live on smooth manifolds. The concept of a tensor field is independent of any coordinate system: changing the coordinate system has no effect on the tensor field itself, even though it changes the way the tensor field is represented in terms of coordinates. In contrast, given any smooth rearrangement of the manifold's points (that is, any diffeomorphism from the manifold to itself), a corresponding transformation of tensor fields may be defined in a natural way. This article defines that transformation and calls it a **fieldomorphism**. Unlike a coordinate transformation, a fieldomorphism *does* change the tensor fields. In general relativity, any fieldomorphism of a solution of the equations of motion gives another solution of the equations of motion, a property often called **general covariance**. This is a **gauge symmetry** in general relativity, which means that two solutions related by a fieldomorphism are physically equivalent to each other.

Contents

1 Introduction

Article 09894 introduced the concept of a **tensor field** on a smooth manifold. Tensor fields are defined independently of any coordinate system, even though we typically use a coordinate representation to describe them. Given any **diffeomorphism** from the underlying smooth manifold to itself,[1] this article defines a corresponding transformation of tensor fields. Physicists often call this a diffeomorphism, but mathematicians usually reserve that name only for the transformation of the underlying smooth manifold, so this article uses the name **fieldomorphism** for the corresponding transformation of the tensor fields.[2] In general relativity, fieldomorphisms are symmetries, in the sense that any fieldomorphism of a solution of the equations of motion gives another solution of the equations of motion.[3] This symmetry is often called **general covariance** or **diffeomorphism invariance** in the physics literature.

One of the main messages in this article is the difference between a fieldomorphism (which changes the tensor fields) and a coordinate transformation (which does not). Understanding the difference between fieldomorphisms and coordinate transformations is still clarifying even when fieldomorphism symmetry is a **gauge symmetry** – in other words, even when two solutions related by a fieldomorphism are physically equivalent to each other, like they are in general relativity (section 12).

[1] Lee (2013), reviewed in article 93875

[2] This name is not standard, but at least it doesn't seem to be already in use for anything else: when I searched for the name *fieldomorphism* online before posting these articles, Google said "It looks like there aren't many great matches for your search."

[3] To be a symmetry, the same fieldomorphism must be applied to *all* of the model's fields (and other dynamic entities, if any), including to the metric field.

2 Diffeomorphisms: one-dimensional example

Loosely speaking, a diffeomorphism (from a smooth manifold to itself) is a smooth rearrangement of the manifold's points. Diffeomorphisms preserve the smooth structure, but typically don't preserve geometric structure (like distances, angles, or curvature).

For a simple example, consider the 1-dimensional smooth manifold $\mathbb{R}$, the real line with its usual smooth structure. The map $\sigma : \mathbb{R} \to \mathbb{R}$ defined by

$$\sigma(x) = \sinh x \equiv \frac{e^x - e^{-x}}{2}$$

is a diffeomorphism from $\mathbb{R}$ to itself. The inverse function σ^{-1} exists because $\sigma(x)$ is a monotonically increasing function of x (its derivative is not zero anywhere) and because its range covers all of $\mathbb{R}$:

$$\sigma(x) \to \pm\infty \qquad \text{as } x \to \pm\infty.$$

3 Diffeomorphisms: two-dimensional example

For another example, consider the 2-dimensional smooth manifold $\mathbb{R}^2$ equipped with its usual smooth structure. A point in $\mathbb{R}^2$ is represented by a pair (x, y) of real numbers x and y. We can use x and y as the **coordinates** of the point. Let σ denote any smooth function σ from $\mathbb{R}^2$ to itself, with a smooth inverse:

$$(\tilde{x}, \tilde{y}) = \sigma(x, y) \qquad\qquad (x, y) = \sigma^{-1}(\tilde{x}, \tilde{y}),$$

Saying that σ is smooth means that $\tilde{x}$ and $\tilde{y}$ are both smooth functions of x, y, and saying that the inverse σ^{-1} is smooth means that x and y are both smooth functions of $\tilde{x}, \tilde{y}$. Such a function can be used in either of two ways:

- as a **coordinate transform**, which *relabels* the points of $\mathbb{R}^2$, or...
- as a **diffeomorphism**, which *permutes* the points of $\mathbb{R}^2$.

Example:

$$\begin{aligned} \tilde{x} &= x\,\cos\theta(x, y) + y\,\sin\theta(x, y) \\ \tilde{y} &= y\,\cos\theta(x, y) - x\,\sin\theta(x, y) \end{aligned} \tag{1}$$

and

$$\theta(x, y) \equiv \frac{\pi/2}{1 + x^2 + y^2}.$$

This map is depicted in figure 1. To see that this map has a smooth inverse, notice that (1) implies

$$\tilde{x}^2 + \tilde{y}^2 = x^2 + y^2 \qquad \Rightarrow \qquad \theta(\tilde{x}, \tilde{y}) = \theta(x, y).$$

Use this to verify the relationships

$$\begin{aligned} x &= \tilde{x}\,\cos\theta(\tilde{x}, \tilde{y}) - \tilde{y}\,\sin\theta(\tilde{x}, \tilde{y}) \\ y &= \tilde{y}\,\cos\theta(\tilde{x}, \tilde{y}) + \tilde{x}\,\sin\theta(\tilde{x}, \tilde{y}). \end{aligned} \tag{2}$$

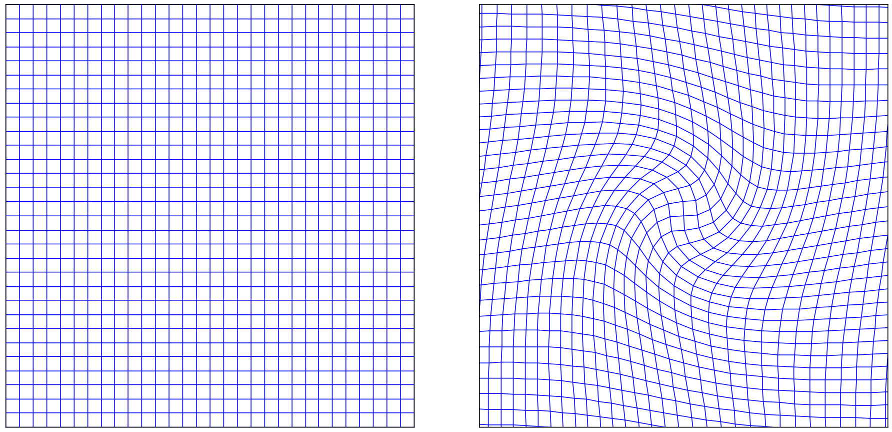

Figure 1 – The left picture shows some lines of constant x and some lines of constant y. The right picture shows some lines of constant $\tilde{x}$ and some lines of constant $\tilde{y}$. The lines of constant x, y were drawn to be straight with respect to the page's inherent metric structure, but we could have drawn the lines of constant $\tilde{x}, \tilde{y}$ to be straight instead. On a smooth manifold without a metric structure, straightness is undefined, so neither coordinate system is more natural than the other.

This demonstrates that the smooth map (1) has a smooth inverse. Using $(x, y) = (1, 1)$ in equation (1) gives $(\tilde{x}, \tilde{y}) = \left(\frac{\sqrt{3}+1}{2}, \frac{\sqrt{3}-1}{2}\right)$. We can use (1) in either of two ways:

- As a **coordinate transform:** we can regard the map as giving a new name $\left(\frac{\sqrt{3}+1}{2}, \frac{\sqrt{3}-1}{2}\right)$ to the point that was previously named $(1, 1)$.

- As a **diffeomorphism:** we can regard the map as moving the point with coordinates $(1, 1)$ to the new location with coordinates $\left(\frac{\sqrt{3}+1}{2}, \frac{\sqrt{3}-1}{2}\right)$.

4 Notation and basic identities

Let $x = (x^1, x^2, ..., x^N)$ denote the coordinates of a point in the N-dimensional smooth manifold $\mathbb{R}^N$, and let $\tilde{x} = (\tilde{x}^1, \tilde{x}^2, ..., \tilde{x}^N)$ be related to x by a smooth map with a smooth inverse. The rest of this article uses the abbreviations

$$\partial_a \equiv \frac{\partial}{\partial x^a} \qquad\qquad \tilde{\partial}_a \equiv \frac{\partial}{\partial \tilde{x}^a}.$$

In more detail, for any given value of the index a:

- ∂_a is the derivative with respect to x^a, with x^b held fixed for all $b \neq a$.
- $\tilde{\partial}_a$ is the derivative with respect to $\tilde{x}^a$, with $\tilde{x}^b$ held fixed for all $b \neq a$.

The identities[4]

$$d\tilde{x}^a = \big(\partial_\bullet \tilde{x}^a\big)\, dx^\bullet \qquad\qquad \tilde{\partial}_a = \big(\tilde{\partial}_a x^\bullet\big)\, \partial_\bullet$$

$$dx^a = \big(\tilde{\partial}_\bullet x^a\big)\, d\tilde{x}^\bullet \qquad\qquad \partial_a = \big(\partial_a \tilde{x}^\bullet\big)\, \tilde{\partial}_\bullet$$

$$d\tilde{x}^\bullet\, \tilde{\partial}_\bullet = dx^\bullet\, \partial_\bullet$$

$$\big(\tilde{\partial}_a x^\bullet\big)\big(\partial_\bullet \tilde{x}^b\big) = \delta_a^b = \big(\partial_a \tilde{x}^\bullet\big)\big(\tilde{\partial}_\bullet x^b\big)$$

(with implied sums over the index $\bullet$) will be used frequently.

[4] I occasionally use a non-alphanumeric character (like $\bullet$) for an index because this can help make the pattern stand out more clearly. This isn't standard practice, but it's mild compared to the graphic notation advocated in Penrose (1971), reviewed later in Penrose (1984), and used extensively in Cvitanović (2011). Similar graphic notations have also been used in other contexts, including quantum physics: examples include Coecke *et al* (2021) and Biamonte and Bergholm (2017).

5 Fieldomorphisms

A fieldomorphism is *neither* a coordinate transform (which merely relabels things) *nor* a diffeomorphism (which rearranges the points of the smooth manifold). Instead, a **fieldomorphism** rearranges the set of tensor fields of each type, *without* rearranging (or relabeling) the points of the smooth manifold.[5]

Consider any smooth map

$$\sigma : \mathbb{R}^N \to \mathbb{R}^N \qquad\qquad \tilde{x} = \sigma(x)$$

with a smooth inverse. The corresponding fieldomorphism replaces each scalar field ϕ with the new scalar field $\tilde{\phi}$ defined by

$$\boxed{\tilde{\phi}(x) \equiv \phi(\tilde{x}).} \tag{3}$$

Intuitively: the fieldomorphism rearranges the relationship between the scalar field's values and the underlying smooth manifold's points – but it does this by morphing the field instead of by morphing the underlying smooth manifold.

The same fieldomorphism affects vector fields in a natural way. Recall[6] that a vector field is a (special kind of) map from the set of scalar fields to itself. A given vector field V maps a given scalar field S to the scalar field $V(S)$. The fieldomorphism replaces V with $\tilde{V}$, which is the vector field that maps $\tilde{S}$ to $\widetilde{V(S)}$:

$$\tilde{V}(\tilde{S}) = \widetilde{V(S)}. \tag{4}$$

In words: the fieldomorphism of V maps the fieldomorphism of S to the fieldomorphism of $V(S)$. The quantities S and $V(S)$ are both scalar fields, so their

[5] Fieldomorphisms are similar to **pullbacks** and **pushforwards** (Lee (2013), illustrated by example 8.20 on pages 183-184), but with a different perspective: pullbacks and pushforwards are used to "transfer" tensor fields from one smooth manifold to another smooth manifold, given a diffeomorphism from one manifold to the other. Fieldomorphisms are used to *replace* the original tensor fields with new tensor fields on the same smooth manifold.

[6] Article 09894

fieldomorphisms $\tilde{S}$ and $\widetilde{V(S)}$ are defined by (3). The condition (4) is required to hold for all scalar fields S, to it defines $\tilde{V}$ unambiguously. More explicitly, using a coordinate representation: if[7]

$$V(S) = V^a(x)\,\partial_a S(x),$$

then

$$\tilde{V}^a(x)\,\partial_a \tilde{S}(x) = \tilde{V}(\tilde{S}) = \widetilde{V(S)} = V^a(\tilde{x})\,\tilde{\partial}_a S(\tilde{x}).$$

This holds for all scalar fields S, so it implies

$$\tilde{V}^a(x)\,\partial_a = V^a(\tilde{x})\,\tilde{\partial}_a. \tag{5}$$

Using an identity from section 4, this may also be written

$$\boxed{\tilde{V}^a(x) = \big(\tilde{\partial}_\bullet x^a\big) V^\bullet(\tilde{x}).} \tag{6}$$

Equation (6) describes how the fieldomorphism affects the componentes of a vector field, just like equation (3) describes how the fieldomorphism affects a scalar field (which has only one component).

The effect of a fieldomorphism on one-form fields is defined similarly. Recall[8] that a one-form field is a (special kind of) map from the set of vector fields to the set of scalar fields: if ω is a one-form field and V is a vector field, then $\omega(V)$ is a scalar field. We've already defined the effect of the fieldomorphism on scalar and vector fields, so we can define its effect on one-form fields by requiring

$$\tilde{\omega}(\tilde{V}) = \widetilde{\omega(V)} \tag{7}$$

for all vector fields V. In a coordinate representation,

$$\tilde{\omega}(\tilde{V}) = \tilde{\omega}_a(x)\tilde{V}^a(x) \qquad\qquad \widetilde{\omega(V)} = \omega_a(\tilde{x})V^a(\tilde{x})$$

[7] Recall (article 09894) that in a coordinate representation, a vector field is a derivative operator.

[8] Article 09894

(the second equation is an application of equation (3)), so equations (6) and (7) imply

$$\boxed{\tilde{\omega}_a(x) \equiv \big(\partial_a \tilde{x}^\bullet\big)\omega_\bullet(\tilde{x}).} \tag{8}$$

after using an identity from section 4. This can also be written

$$\tilde{\omega}_a(x)dx^a \equiv \omega_a(\tilde{x})d\tilde{x}^a. \tag{9}$$

Equation (8) (or (9)) describes how the fieldomorphism affects the components of a one-form field, just like equation (6) (or (5)) describes how the fieldomorphism affects the components of a vector field.

The effect of a fieldomorphism on other tensor fields is defined similarly. For a tensor field with components $T_{\cdots}^{\cdots}(x)$, the result is that the components of $\tilde{T}_{\cdots}^{\cdots}$ are $T_{\cdots}^{\cdots}(\tilde{x})$ contracted with one factor of $\partial x/\partial\tilde{x}$ for each upper index and with one factor of $\partial\tilde{x}/\partial x$ for each lower index. Example:

$$\boxed{\tilde{T}_{ab}(x) \equiv \big(\partial_a \tilde{x}^\bullet\big)\big(\partial_b \tilde{x}^\times\big)T_{\bullet\times}(\tilde{x}),} \tag{10}$$

which can also be written

$$\tilde{T}_{ab}(x)\,dx^a \otimes dx^b \equiv T_{ab}(\tilde{x})\,d\tilde{x}^a \otimes d\tilde{x}^b. \tag{11}$$

An important message here is that a fieldomorphism affects *only the fields*, not the coordinates and not the points of the underlying manifold.

6 Fieldomorphisms: one-dimensional examples

For some easy examples, consider this smooth map from the one-dimensional smooth manifold $\mathbb{R}$ to itself:

$$\tilde{x} = \sigma(x) = \sinh x. \tag{12}$$

This map has a smooth inverse, because the derivative $d\tilde{x}/dx$ is not zero anywhere. The effect of the corresponding fieldomorphism on various types of tensor field is illustrated by these examples:[9]

- For a scalar field ϕ, the morphed scalar field is given by equation (3):

  $$\tilde{\phi}(x) = \phi(\sinh x).$$

  In particular, if the original scalar field is $\phi(x) = x^2$, then the transformed scalar field is $\tilde{\phi}(x) = (\sinh x)^2$.

- For a vector field with component $V^a(x)$, the component of the morphed vector field is given by equation (5):[10]

  $$\tilde{V}^a(x) = \frac{V^a(\sinh x)}{\cosh x}.$$

- For a one-form field with component $\omega_a(x)$, the component of the morphed field is given by equation (9):

  $$\tilde{\omega}_a(x) = (\cosh x)\,\omega_a(\sinh x).$$

- For a metric field with component $g_{ab}(x)$, the component of the morphed metric field is given by equation (11):

  $$\tilde{g}_{ab}(x) = (\cosh x)^2\, g_{ab}(\sinh x).$$

[9] To check these results, use the identities listed in section 4.

[10] The underlying smooth manifold is only one-dimensional, so a vector field has only one component (as does any other type of tensor field in this case), but it is still *not* a scalar field! I'm retaining the index as a reminder of this.

7 Fieldomorphisms: two-dimensional example

For an example with multi-component tensor fields, consider this smooth map from the two-dimensional smooth manifold $\mathbb{R}^2$ to itself (using matrix notation):

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

This map has a smooth inverse, namely

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}.$$

The effect of the corresponding fieldomorphism on various types of tensor field is illustrated by these examples:

- For a scalar field ϕ, the morphed scalar field is given by equation (3):

  $$\tilde{\phi}(x,y) = \phi(5x+7y,\, 2x+3y).$$

  In particular, if the original scalar field is $\phi(x,y) = x^2 y$, then the transformed scalar field is $\tilde{\phi}(x,y) = (5x+7y)^2(2x+3y)$.

- For a vector field with components $V^a(x)$, the components of the morphed vector field are given by equation (6):

  $$\begin{pmatrix} \tilde{V}^x(x,y) \\ \tilde{V}^y(x,y) \end{pmatrix} = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} \begin{pmatrix} V^x(5x+7y,\, 2x+3y) \\ V^y(5x+7y,\, 2x+3y) \end{pmatrix}.$$

8 Fieldomorphisms and the EM gauge field, part 1

As explained in article 98002, when the smooth manifold is topologically trivial, the tensor F_{ab} describing the electromagnetic (EM) field can also be written in terms of a gauge field A_a:

$$F_{ab}(x) = \partial_a A_b(x) - \partial_b A_a(x). \tag{13}$$

The left- and right-hand sides of equation (13) are supposed to be two different ways of representing the same thing, but are they affected the same way by a fieldomorphism? In other words, does (13) imply

$$\tilde{F}_{ab}(x) = \partial_a \tilde{A}_b(x) - \partial_b \tilde{A}_a(x) \tag{14}$$

for all fieldomorphisms?

Remember that a fieldomorphism does *not* affect the coordinates, which is why equations (13) and (14) both involve ∂ instead of $\tilde{\partial}$ on the right-hand side. Even so, equation (13) actually does imply equation (14). This might not be obvious, and it wouldn't be true if the minus-sign on the right-hand side were changed to a plus-sign! The minus-sign is essential if both F_{ab} and A_a are to qualify as *tensor fields*, as assumed by the definition of *fieldomorphism*. The next section shows that (13) implies (14), highlighting the essential role of the minus sign.

9 Fieldomorphisms and the EM gauge field, part 2

To see that (13) implies (14), start with

$$\tilde{F}_{ab}(x) \equiv \big(\partial_a\tilde{x}^\bullet\big)\big(\partial_b\tilde{x}^\times\big)F_{\bullet\times}(\tilde{x}) \qquad\qquad \tilde{A}_a(x) \equiv \big(\partial_a\tilde{x}^\bullet\big)A_\bullet(\tilde{x}), \tag{15}$$

which describe how a fieldomorphism affects the tensor fields F and A. Equation (13) says[11]

$$F_{\bullet\times}(\tilde{x}) = \tilde{\partial}_\bullet A_\times(\tilde{x}) - \tilde{\partial}_\times A_\bullet(\tilde{x}).$$

Combine this with the first of equations (15) to get

$$\begin{aligned}\tilde{F}_{ab}(x) &= \big(\partial_a\tilde{x}^\bullet\big)\big(\partial_b\tilde{x}^\times\big)\big(\tilde{\partial}_\bullet A_\times(\tilde{x}) - \tilde{\partial}_\times A_\bullet(\tilde{x})\big)\\ &= \big(\partial_b\tilde{x}^\times\big)\partial_a A_\times(\tilde{x}) - \big(\partial_a\tilde{x}^\bullet\big)\partial_b A_\bullet(\tilde{x}),\end{aligned} \tag{16}$$

after using an identity from section 4 to relate $\tilde{\partial}$ to ∂. This is the left-hand side of (14). For the right-hand side, the second of equations (15) implies

$$\begin{aligned}\partial_a\tilde{A}_b(x) - \partial_b\tilde{A}_a(x) &= \partial_a\Big(\big(\partial_b\tilde{x}^\bullet\big)\,A_\bullet(\tilde{x})\Big) - \partial_b\Big(\big(\partial_a\tilde{x}^\bullet\big)\,A_\bullet(\tilde{x})\Big)\\ &= \big(\partial_b\tilde{x}^\bullet\big)\partial_a A_\bullet(\tilde{x}) - \big(\partial_a\tilde{x}^\bullet\big)\partial_b A_\bullet(\tilde{x})\\ &\quad + \left(\partial_a\partial_b\tilde{x}^\bullet - \partial_b\partial_a\tilde{x}^\bullet\right)A_\bullet(\tilde{x}).\end{aligned}$$

Thanks to the minus sign, the last line is zero because partial derivatives with respect to independent variables commute with each other, and the remainder is equal to (16). This completes the proof that (13) implies (14), for all fieldomorphisms.

[11] The derivatives here are $\tilde{\partial}$ because this is dictated by the argument $\tilde{x}$ on the left-hand side. This isn't a fieldomorphism or a coordinate transform. It's just a matter of using $\tilde{x}$ as the argument of $F_{\bullet\times}(\cdots)$ according to equation (13).

10 Fieldomorphisms as symmetries: concept

General relativity has a remarkable property called **general covariance**: given any solution of the equations of motion, applying any fieldomorphism to all of the fields (and particle-worldlines, if any) gives another solution to the same equations of motion. This is a *huge* symmetry, and most models don't have it. Every model has the property that its equations (and their solutions) can be expressed in any coordinate system, but that's trivial, because changing the coordinate system merely relabels things. In contrast, general covariance is not a trivial property at all.

To make the math easier, the next section considers a model – namely Maxwell's equations in flat spacetime – that is symmetric under *some* fieldomorphisms, namely those that leave the metric field unchanged.[12] Fieldomorphisms that don't change the metric field are called **isometries**.[13] When the metric field is the Minkowski metric, as it will be in this example, isometries are called **Poincaré transforms**. This includes **Lorentz transforms**, which are the focus of the example in the next section. A model that is symmetric under these fieldomorphisms is said to have **Lorentz symmetry**. This is a much smaller symmetry than general covariance,[14] but it still has important applications and still illustrates some important concepts.

Article 49705 illustrated the concept of Lorentz symmetry in models with only scalar fields. The following example uses electrodynamics instead. This makes things more interesting, because it illustrates the importance of the factors $\partial x/\partial\tilde{x}$ and $\partial\tilde{x}/\partial x$ that a fieldomorphism applies to the components of non-scalar fields.

[12] Maxwell's equations can be generalized to be symmetric under all fieldomorphisms by replacing partial derivatives with covariant derivatives and promoting the metric field to a dynamic field subject to the equation of motion $R_{abc}{}^{d} = 0$, where $R_{abc}{}^{d}$ is the Riemann curvature tensor (article 03519). That model doesn't satisfy the action principle (because the EM field still doesn't influence the metric field), but it is does have full general covariance.

[13] An isometry can also be defined as a diffeomorphism for which the pullback of the metric field is the same as the original metric field.

[14] Here, I'm not distinguishing between gauge symmetries (section 1) and non-gauge symmetries. General covariance is a gauge symmetry, but Lorentz symmetry is not usually called a gauge symmetry – even though applying a Lorentz transform to *everything* in a flat universe would obviously not have any observable effect. Using words with perfect consistency is practically impossible whenever we try to use the same words in a variety of different models that have different degrees of "fundamentalness."

11 Fieldomorphisms as symmetries: example

Consider Maxwell's equations for the electromagnetic (EM) field in flat spacetime. The EM field is a tensor field with components F_{ab}. In flat spacetime with the Minkowski metric, Maxwell's equations are (article 31738)

$$\partial_a F^{ab}(x) = 0 \qquad\qquad \partial_{[a} F_{bc]}(x) = 0 \tag{17}$$

where indices are raised/lowered using the Minkowski metric (with components η_{ab}) and its inverse (with components η^{ab}). Square brackets around the subscripts denote complete antisymmetrization.

A Lorentz transform is a fieldomorphism for which the quantities $\partial\tilde{x}^a/\partial x^b$ satisfy

$$\partial_a\big(\partial_b\tilde{x}^c\big) = 0 \tag{18}$$

$$\big(\partial_\bullet\tilde{x}^a\big)\big(\partial_\times\tilde{x}^b\big)\eta^{\bullet\times} = \eta^{ab} \tag{19}$$

with implied sums over the indices $\bullet$ and $\times$. The goal is to demonstrate that such fieldomorphisms are symmetries of Maxwell's equations (17): if F satisfies equations (17), then $\tilde{F}$ does, too.

As explained in section 5, the components of $\tilde{F}$ are related to those of F by

$$\tilde{F}_{ab}(x) \equiv \big(\partial_a\tilde{x}^\bullet\big)\big(\partial_b\tilde{x}^\times\big)F_{\bullet\times}(\tilde{x}) \tag{20}$$

$$\tilde{F}^{ab}(x) \equiv \big(\tilde{\partial}_\bullet x^a\big)\big(\tilde{\partial}_\times x^b\big)F^{\bullet\times}(\tilde{x}). \tag{21}$$

The goal is to show that the quantities

$$\partial_a\tilde{F}^{ab}(x) \qquad\qquad \partial_{[a}\tilde{F}_{bc]}(x) \tag{22}$$

are both zero if F satisfies (17). Use (18) and (20)-(21) to get

$$\partial_a\tilde{F}^{ab}(x) = \big(\tilde{\partial}_\bullet x^a\big)\big(\tilde{\partial}_\times x^b\big)\partial_a F^{\bullet\times}(\tilde{x}) \qquad\qquad \partial_{[a}\tilde{F}_{bc]}(x) = \big(\partial_{[b}\tilde{x}^\bullet\big)\big(\partial_c\tilde{x}^\times\big)\partial_{a]}F_{\bullet\times}(\tilde{x}).$$

The condition (18) was used to pull the factors $\partial x/\partial\tilde{x}$ and $\partial\tilde{x}/\partial x$ out from under the derivative ∂_a. Use an identity from section 4 to express ∂_a in terms of $\tilde{\partial}_a$, which gives

$$\begin{aligned}\partial_a\tilde{F}^{ab}(x) &= \big(\partial_a\tilde{x}^\circ\big)\big(\tilde{\partial}_\bullet x^a\big)\big(\tilde{\partial}_\times x^b\big)\tilde{\partial}_\circ F^{\bullet\times}(\tilde{x})\\ &= \big(\tilde{\partial}_\times x^b\big)\big[\tilde{\partial}_a F^{a\times}(\tilde{x})\big]\end{aligned} \tag{23}$$

$$\partial_{[a}\tilde{F}_{bc]}(x) = \big(\partial_a\tilde{x}^\circ\big)\big(\partial_b\tilde{x}^\bullet\big)\big(\partial_c\tilde{x}^\times\big)\big[\tilde{\partial}_{[\circ}F_{\bullet\times]}(\tilde{x})\big]. \tag{24}$$

In equation (23), the factors involving the index a canceled thanks to another identity from section 4. The quantities in square brackets are just $\partial_a F^{ab}$ and $\partial_{[a}F_{bc]}$ with the coordinates/indices re-labeled, so they are zero if F satisfies (17). Altogether, this confirms that if F satisfies (17), then so does $\tilde{F}$, whenever the fieldomorphism is a Lorentz transform.

Notice how the derivation used the conditions (18) and (19):

- The condition (18) was used explicitly to pull the factors $\partial x/\partial\tilde{x}$ and $\partial\tilde{x}/\partial x$ out from under the derivative ∂_a. Fieldomorphisms that don't satisfy (18) are not symmetries of the pair of equations (17), but *all* fieldomorphisms are symmetries of the equation $\partial_{[a}F_{bc]} = 0$ by itself. The antisymmetrization is essential for this, as it was in section 9.

- We used the condition (19) implicitly when we declared that indices are raised/lowered with the Minkowski metric. For any metric field g whatsoever, we always have the relationship $F^{ab} = g^{a\bullet}g^{b\times}F_{\bullet\times}$. With this relationship, equation (20) implies equation (21), as long as the same fieldomorphism is applied to the metric, too. Most fieldomorphisms change the metric field, though, so if we want equations (20)-(21) to be consistent with using the *Minkowski* metric to raise/lower indices, then we need to limit the fieldomorphisms to those that satisfy the condition (19).

12 General covariance as a gauge symmetry

The previous section illustrated the idea of fieldomorphism symmetry, using a model for which some fieldomorphisms are symmetries. In general relativity, all fieldomorphisms are symmetries – a property called general covariance. This section illustrates the interpretation of fieldomorphisms as *gauge* symmetries in general relativity. A fieldomorphism is called a **gauge symmetry** if any two solutions related to each other by the fieldomorphism are considered to be physically equivalent to each other. In a model like general relativity, where *all* fieldomorphisms are gauge symmetries, observables must be invariant under all fieldomorphisms. Examples:

- Consider a model whose dynamic entities include a scalar field. The question "what is the field's maximum value?" is an observable, but the question "at what point in spacetime does the maximum value occur?" is not.

- For a model that includes two scalar fields, the question "do the two fields attain their maximum values at the same point or at different points?" is an observable, but the question "at what points do the fields attain their maximum values?" is not.

- For a model with a scalar field and a metric field, the question "what are the values of the scalar field at points where the metric field's curvature is zero?" is an observable, but the question "at what points of the smooth manifold does the metric field have zero curvature?" is not.

Roughly speaking, in a model like general relativity in which all fieldomorphisms are gauge symmetries, observables are *relationships between dynamic entities*. Fieldomorphisms preserve those relationships, because all of the fields are "morphed" together. If the model has pointlike particles, then the gauge symmetries "morph" the particles' worldlines together with the fields (including the metric field), and then the question "is the particle's worldline a geodesic?" is an observable. This is related to **Mach's principle**.

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